Boundary Harnack inequalities for regional fractional Laplacian[1]

Qing-Yang Guan [2]

Abstract

We consider boundary Harnack inequalities for regional fractional Laplacian which are generators of censored stable-like processes on G taking

$$\kappa(x,y)/|x-y|^{n+\alpha}dxdy, \quad x,y\in G$$

as the jumping measure. When G is a $C^{1,\beta-1}$ open set, $1<\alpha<\beta\leq 2$ and $\kappa\in C^1(\overline{G}\times\overline{G})$ bounded between two positive numbers, we prove a boundary Harnack inequality giving $dist(x,\partial G)^{\alpha-1}$ order decay for harmonic functions near the boundary. For a $C^{1,\beta-1}$ open set $D\subset\overline{D}\subset G$, $0<\alpha\leq(1\vee\alpha)<\beta\leq 2$, we prove a boundary Harnack inequality giving $dist(x,\partial D)^{\alpha/2}$ order decay for harmonic functions near the boundary. These inequalities are generalizations of the known results for the homogeneous case on $C^{1,1}$ open sets. We also prove the boundary Harnack inequality for regional fractional Laplacian on Lipschitz domain.

Key words fractional Laplacian, symmetric α-stable processes, censored stable processes, boundary Harnack inequality
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1 Introduction

Let G be an open set in $\mathbb{R}^n$ and κ a positive symmetric function on $\overline{G}\times\overline{G}$. For $0<\alpha<2$, the regional fractional (fractional-like) Laplacian is defined by

$$\Delta_{G}^{\frac{\alpha}{2},\kappa}u(x)=\lim_{\varepsilon\downarrow 0}\mathcal{A}(n,-\alpha)\int_{y\in G,|y-x|>\varepsilon}\frac{\kappa(x,y)(u(y)-u(x))}{|x-y|^{n+\alpha}}\,dy,\quad x\in\overline{G}, \tag{1.1}$$

provided the limit exists, see [23]. Here $\mathcal{A}(n,-\alpha)=|\alpha|2^{\alpha-1}\Gamma((n+\alpha)/2)\pi^{-n/2}/\Gamma(1-\alpha/2)$ coming from $\Delta_{\mathbb{R}^n}^{\frac{\alpha}{2},\kappa}=-(-\Delta)^{\alpha/2}$ when $\kappa\equiv 1$ and we refer to Guan and Ma [24] for $\kappa\equiv 1$ in (1.1). Under some regularity conditions, it is known that the $\alpha/2$ power of a second order elliptic operator with Neumann boundary condition is an example of (1.1). Since the integral kernel in (1.1) may not be homogeneous in space, these operators to fractional Laplacian are similar to the second order elliptic operators to Laplacian. For $1<\alpha<2$, among others, an explicit boundary Harnack inequality (BHI) for $\Delta_{G}^{\frac{\alpha}{2},1}$ was proved in Bogdan, Burdzy and Chen [12] on $C^{1,1}$ open sets, where it is called the BHI of the censored stable processes. The main aim of this paper is to consider the same type inequality for the nonhomogeneous case and the corresponding BHI on Lipschitz domain.

Boundary Harnack inequalities are important tools in studying the boundary value problems in partial differential equations and potential theory of Markov processes. Analytically, such inequalities describe an uniform asymptotic behavior for solutions of the Dirichlet problems near the boundary. In Chen and Kim [14], the BHI in [12] was used in the proof of the Green function estimates of censored stable processes. See also Bogdan [9] for the Brownian motion case. We refer to Bass [2], Chen, Kim and Song [15] [16] for more applications of the BHI.

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[2] Institute of Applied Mathematics, Academy of Mathematics and Systems Science, Chinese Academy of Sciences, email: guanqy@amt.ac.cn

Boundary Harnack inequalities were first proved for Laplacian in Dahlberg [19] and Ancona [1] on Lipschitz domains. It was later extended to the second order elliptic operators in divergence form in Caffarelli, Fabes, Mortola and Salsa [13], and in nondivergence form in Fabes, Garofalo, Marín-Malave and Salsa [20]. A probability method for studying such inequalities began in Bass and Burdzy [4]. This method was applied to prove the BHI for Laplacian on Hölder domains for elliptic operators in divergence form in Banuelos, Bass and Burdzy [3].

The study of the BHI for the fractional Laplacian began in Bogdan [8], Bogdan and Byczkowski [10] on Lipschitz open sets. Significant progresses have been made on open sets in Song and Wu [28] and the recent paper Bogdan, Kulczycki and Kwasnicki [11]. An explicit BHI for the fractional Laplacian was first given in Chen and Song [18] on $C^{1,1}$ open sets. Due to the jumps of stable processes or equivalently the nonlocal property of their generators, the corresponding harmonic functions show different feathers from the Laplacian case. Compared with the fractional Laplacian case in [18] ($0 < \alpha < 2$), the BHI in [12] for the regional fractional Laplacian ($1 < \alpha < 2$) gives a different decay for harmonic functions near the boundary, i.e., the former is of order $\rho(x)^{\alpha/2}$ and the later is of order $\rho(x)^{\alpha-1}$. In [12], the Markov processes associated with the regional fractional Laplacian under the Dirichlet boundary condition were first introduced and called the censored stable processes. We refer to [12] for some other characterizations of this process.

A standard box method to prove the BHI includes comparison of harmonic measures around boxes, the Harnack inequality and the Carleson estimate. We refer to Bass and Burdzy [4] for this method in the diffusion case. The proof of the BHI in [12] studied these steps mainly by explicit harmonic functions given in the same paper and a relation between the censored stable processes and the symmetric α-stable processes. Some strong techniques are involved in this original proof. Due to the importance of this inequality, it is helpful to simplify the proof in [12] and to study this result in more general situations. In particular, our arguments can be used to study the Lipschitz case which is an open problem in this direction. We remark that the (super sub) harmonic functions given in [12] plays a fundamental role in this paper.

To further introduce the results and the methods of this paper, we prepare some definitions below. For $x = (x_1, \cdots, x_n) \in \mathbb{R}^n$, we write $x = (\widetilde{x}, x_n)$. Let $0 < \gamma \leq 1$ and $\Gamma : \mathbb{R}^{n-1} \to \mathbb{R}$. We say that Γ is a $C^{1,\gamma}$ function if it is differentiable and

$$\|\Gamma\|_{1,\gamma} := \sup_{\widetilde{y}\neq\widetilde{x}, |\widetilde{y}-\widetilde{x}|\leq 2} \frac{|\nabla\Gamma(\widetilde{y}) - \nabla\Gamma(\widetilde{x})|}{|\widetilde{y} - \widetilde{x}|^{\gamma}} < \infty, \tag{1.2}$$

where $\nabla = (\partial/\partial x_i)_{i=1}^{n-1}$. The constant 2 in (1.2) is only for the convenience of the later use. Let G be an open set in $\mathbb{R}^n$. We say that G is a special $C^{1,\gamma}$ domain if for some $C^{1,\gamma}$ function $\Gamma : \mathbb{R}^{n-1} \to \mathbb{R}$, G can be represented as $\{x = (\widetilde{x}, x_n) \in \mathbb{R}^n, x_n > \Gamma(\widetilde{x})\}$. In this case G is also denoted by G_Γ. We say that G is $C^{1,\beta-1}$ if there exist $r_0 > 0$ and $\Lambda > 0$ such that for each $z \in \partial G$, we can find a $C^{1,\beta-1}$ function $\Gamma_z : \mathbb{R}^{n-1} \to \mathbb{R}$ with $\|\Gamma\|_{1,\gamma} \leq \Lambda$ and an orthonormal coordinate system CS_z such that

$$G \cap B\big(z, r_0\big) = \{y = (y_1, \cdots, y_n) :\ y_n > \Gamma_z(y_1, \cdots y_{n-1})\ \} \cap B\big(z, r_0\big). \tag{1.3}$$

By rotation and translation, we can always assume that $\nabla\Gamma_z(\widetilde{z}) = \Gamma_z(\widetilde{z}) = 0$ in CS_z. The pair (r_0, Λ) is called the characteristics of G. The characteristics of a Lipschitz open set is defined in a similar way. For each $\delta > 0$, set

$$G'_\delta = \{y \in G : \rho(y) < \delta\}, \qquad G_\delta = \{y \in G : \rho(y) > \delta\}, \tag{1.4}$$

where $\rho(y) = dist(y, \partial G)$.

Let $1 < \alpha < 2$ and ψ_1, ψ_2 be positive functions in $C^1(\overline{G} \times \overline{G})$. Let κ be a symmetric function on $\overline{G} \times \overline{G}$ taking values between two positive numbers C_1 and C_2. Assume also that for some

constant $C' > 0$ and $\delta \in (0, r_0)$

$$\begin{cases} \left|\kappa(x,y) - \psi_1(x,y) - \psi_2(x,y)\frac{|x-y|^{n+\alpha}}{|x-\overline{y}|^{n+\alpha}}\right| \le C'|x-y| \ , & x, y \in G'_\delta, \\ |\kappa(x,y) - \kappa(x,x)| \le C'|x-y|, & x, y \in G_{\delta/2}, \end{cases} \tag{1.5}$$

where $\overline{y}$ is the reflection point of y with respect to ∂G (see section 4). Write

$$M := C' + \sup_{x,y\in\overline{G},|x-y|<1} (|\nabla_y \psi_1(x,y)| + |\nabla_y \psi_2(x,y)|)$$

and denote by (X_t) the reflected stable-like process. The following theorem is an extension of the BHI in [12] for $\kappa \equiv 1$ on $C^{1,1}$ domain.

Theorem 1.1. *Assume that α, κ satisfy all the conditions above and G is a $C^{1,1}$ open set in $\mathbb{R}^n$ with characteristics $r_0 \le 1$ and Λ. Let $Q \in \partial G$ and $r \in (0, r_0)$. Assume that $u \ge 0$ is a function on G which is not identical to zero, harmonic on $G \cap B(Q,r)$ and vanishes continuously on $\partial G \cap B(Q,r)$. Then there is a constant $C = C(n, \alpha, \Lambda, \delta, C_1, C_2, M)$ such that*

$$\frac{u(x)}{u(y)} \le C\frac{\rho(x)^{\alpha-1}}{\rho(y)^{\alpha-1}}, \qquad x, y \in G \cap B(Q, r/2). \tag{1.6}$$

Moreover, if $\psi_2 \equiv 0$ in (1.6), this boundary Harnack inequality holds for $C^{1,\beta-1}$ open sets with $1 < \alpha < \beta \le 2$.

Here the notation $C = C(n, \alpha, \Lambda, \delta_1, C_1, C_2, M)$ means that the constant C is positive and depends only on parameters in the bracket. This convention will be used throughout the paper. When $\psi_2 \equiv 0$ in (1.6), the last conclusion in Theorem 1.1 was conjectured in [12]. We remark that $\beta = \alpha$ is the critical value in our proof and Theorem 1.1 may not hold for this value.

In [12], when G is a special $C^{1,1}$ domain and $\kappa \equiv 1$, to get sharp estimates of harmonic measures, (super) subharmonic functions are constructed by explicit harmonic functions on $\mathbb{R}^n_+$ and non-explicit perturbations defined through the symmetric α-stable process on $\mathbb{R}^n$. Here we construct explicit (super) subharmonic functions by perturbation directly. This construction may also be used to prove the known explicit BHI for Laplacian, i.e., $\alpha = \beta = 2$ in (1.6).

For the Harnack inequality, we may adopt the method in Bass and Levin [6]. Here we give another proof which might be more straightforward for these nonlocal operators. This proof is similar to the proof of the Carleson estimate given in Lemma 4.2 which is an application of the box method for jump processes taking (4.16) as the key observation . We remark that the method in [6] can be applied to prove the Harnack inequality for jump diffusions, see Song and Vondracek [30]. Therefore by applying the method in this paper, we may prove the BHI for operators like $\Delta + \Delta^{\alpha/2}$ on $C^{1,1}$ open sets, where the decay is of order $\rho(x)$ near the boundary.

The BHI for the fractional Laplacian on $C^{1,1}$ open sets was proved by Poisson kernel estimates in [18]. This and many other estimates of the symmetric stable processes given before depend on their explicit Poisson kernel and Green function for a unit ball which are not available for the nonhomogeneous case. In Lemma 6.1, we present explicit (super, sub) harmonic functions of the fractional Laplacian on half spaces which allow us to study the nonhomogeneous case. See Theorem 6.4. We notice that the harmonic function in Lemma 6.1 has been obtained in Banuelos and Bogdan [7]. As applications, for the fractional-like Laplacian under condition (6.7), we may get the sharp estimates of their Green function and Poisson kernel as in [18] and hence we may get the BHI in [11] under the same conditions.

Another main result of this paper is the following boundary Harnack inequality on Lipschitz domain. The strategy of the proof is essentially the same as the proof of Theorem 1.1.

Theorem 1.2. *Let G be a Lipschitz open set in $\mathbb{R}^n$ with characteristics $r_0 \le 1$ and Λ. Assume that $1 < \alpha < 2$ and κ be a $C^1(\overline{G} \times \overline{G})$ function bounded between two positive numbers. Let $Q \in \partial G$ and $r \in (0, r_0)$. Then there is a constant C such that*

$$\frac{u(x)}{u(y)} \le C \frac{v(x)}{v(y)}, \qquad x, y \in G \cap B(Q, r/2), \tag{1.7}$$

where $u, v \ge 0$ are functions on G which is not identical to zero, harmonic on $G \cap B(Q, r)$ and vanishes continuously on $\partial G \cap B(Q, r)$.

To prove Theorem 1.2, the heat kernel estimate of the reflected stable processes in Chen and Kumagai [17] is used to give some hitting probability estimate. The censored stable processes can be extended to the reflected processes on $\overline{G}$ which is formulated in [12]. For general κ, these two processes are called the censored stable-like process and the reflected stable-like process respectively (see Remark 2.4 [12]). They are symmetric Markov processes on G and $\overline{G}$, respectively. The Dirichlet form of the reflected stable-like process is

$$\begin{aligned} \mathcal{E}^\kappa(u, v) &= \frac{1}{2}\mathcal{A}(n, -\alpha) \int\int_{\overline{G}\times\overline{G}} \frac{\kappa(x, y)(u(x) - u(y))(v(x) - v(y))}{|x - y|^{n+\alpha}}\, dxdy, \\ \mathcal{F}^\kappa &= \{u \in L^2(\overline{G}) : \quad \mathcal{E}^\kappa(u, u) < \infty\}, \end{aligned} \tag{1.8}$$

where $\kappa(x, y)$ is bounded between two positive numbers and G is Lipschitz. In [17], this reflected process was refined to be a Feller process (X_t) on $\overline{G}$ under a more general condition.

The structure of this paper is the following. In Section 2 we study (super, sub) harmonic functions. In Section 3 we prove the Harnack inequality. In Sections 4 and 6 we prove boundary Harnack inequalities for the regional fractional-like Laplacian and the fractional-like Laplacian, respectively. The Lipschitz case is studied in section 5. For $a, b \in \mathbb{R}$, $a \vee b := \max\{a, b\}$. We use $m(\cdot)$ to denote the area measure of $(n-1)$-dimensional subset. For any set U, denote $\tau_U = \inf\{t > 0 : X_t \notin U\}$. For function u on $\mathbb{R}^n$, we always take it as a function on G by restriction when considering $\Delta_G^{\alpha/2} u$. The dimension n is assumed bigger than two throughout the paper.

2 (Super) subharmonic functions for regional fractional Laplacian, $\kappa \equiv 1$

Let u be a Borel function on G and let U be an open subset of G. We say that u is a (super, sub) harmonic function on U with respect to $\Delta_G^{\frac{\alpha}{2},\kappa}$ if $\Delta_G^{\frac{\alpha}{2},\kappa} u(x)(\le, \ge) = 0$ for $x \in U$. We say that u is a (super, sub) harmonic function on U with respect to the reflected stable-like process (X_t) if

$$u(x)(\ge, \le) = E_x u(X_{\tau_B}), \quad x \in B \tag{2.1}$$

for any bounded open set B with $\overline{B} \subset U$, where $\tau_B = \inf\{t > 0 : X_t \notin B\}$. Under the conditions of κ and G in this paper, the harmonic function for the reflected stable-like process is continuous on U (see Corollary 3.6 below). This implies that it is harmonic for $\Delta_G^{\frac{\alpha}{2},\kappa}$ in the weak sense (cf. Theorem 6.6 [25] for $\kappa \equiv 1$). When u is C^2 and κ is C^1, Theorem 4.8 in [23] shows that these two definitions are equivalent. In what follows we use (2.1) for the definition of harmonic functions. We write $\Delta_G^{\frac{\alpha}{2},1}$ by $\Delta_G^{\alpha/2}$ when $x \in G$.

In [12], to establish the BHI for $\kappa \equiv 1$, the following estimates are given for the regional fractional Laplacian acting on function $u = \rho^{\alpha-1}$:

$$\begin{cases} \Delta_G^{\alpha/2} u(x) \le A\rho(x)^{\beta-2}, & x \in G'_{1/A}, \quad if\ \beta < 2, \\ |\Delta_G^{\alpha/2} u(x)| \le A|\ln \rho(x)|, & x \in G'_{1/A}, \quad if\ \beta = 2, \end{cases} \tag{2.2}$$

where G is a special $C^{1,\beta-1}$ domain and A is a positive constant ($1<\alpha<\beta<2$). When $\beta<2$, we can not get a bound for $\Delta_G^{\alpha/2}u$ because it may take $-\infty$. This is related to the fact that ρ may not be C^1 when ∂G is $C^{1,\beta-1}$ ($\beta<2$). To improve estimates (2.2), we replace the distance function by a "height function" which is equal to $x_n-\Gamma_z(x)$ in a neighborhood of $z\in\partial G$.

Since there is no difference for boundary conditions discussed here when $n=1$, we always assume that $n\geq 2$. Let $w_p(y)=y_n^p$ for $y\in\mathbb{R}^n_+$ and $p\in\mathbb{R}$. Our starting point is the following explicit harmonic functions and (2.16) below given in (5.4) [12]

$$\Delta^{\alpha/2}_{\mathbb{R}^n_+} w_{\alpha-1}(x)=0,\quad x\in\mathbb{R}^n_+,\quad \alpha\in(1,2). \tag{2.3}$$

In the following lemma, when necessary, a function defined on a domain is also considered as a function on $\mathbb{R}^n$ by taking zero outside.

Lemma 2.1. *Let $1<\alpha<\beta\leq 2$ and let $\Gamma:\mathbb{R}^{n-1}\to\mathbb{R}$ be a $C^{1,\beta-1}$ function with $\Gamma(\widetilde{0})=0$ and $\nabla\Gamma(\widetilde{0})=0$. Define function $h_{\alpha-1}(x)=(x_n-\Gamma(\widetilde{x}))^{\alpha-1}I_{\{|\widetilde{x}|<2\}}$ for $x\in D:=D_\Gamma$. Then there exists constant $A_1=A_1(n,\alpha,\beta,\|\Gamma\|_{1,\beta-1})$ such that*

$$|\Delta_D^{\alpha/2}h_{\alpha-1}(x)|\leq\begin{cases} A_1\rho(x)^{\beta-2}, & x\in D_1',\ |\widetilde{x}|<1,\quad if\ \beta<2,\\ A_1(|\ln\rho(x)|+1), & x\in D_1',\ |\widetilde{x}|<1,\quad if\ \beta=2.\end{cases} \tag{2.4}$$

Proof Denote $h_{\alpha-1}$ by h. We only prove the lemma for $\alpha<\beta<2$ because the proof for $\beta=2$ is similar. Let $x\in D_1'$ with $|\widetilde{x}|<1$ and choose a point $x_0\in\partial D$ satisfying $\widetilde{x}=\widetilde{x_0}$. Denote by $\overrightarrow{n}(x_0)$ the inward unit normal vector at x_0 for ∂D and set $\Phi(y)=\langle y-x_0,\overrightarrow{n}(x_0)\rangle$ for $y\in\mathbb{R}^n$. It is clear that $\Pi=\{y:\Phi(y)=0\}$ is the plane which is tangent to ∂D at point x_0. Let $\Gamma^*:\widetilde{x}\in\mathbb{R}^{n-1}\to\mathbb{R}$ be the function of plane Π, i.e.,

$$\langle(\widetilde{x},\Gamma^*(\widetilde{x}))-x_0,\overrightarrow{n}(x_0)\rangle=0,$$

and set

$$U=\{y=(\widetilde{y},y_n):y\in D,\ \ |\widetilde{y}-\widetilde{x}|<1,\ \ y_n<2+2^\beta\|\Gamma\|_{1,\beta-1}\ \}.$$

Write $\overline{h}(y)=|y_n-\Gamma^*(\widetilde{y})|$ for $y\in\mathbb{R}^n$. Applying the assumption that ∂G is $C^{1,\beta-1}$ and $\nabla\Gamma(\widetilde{x})-\nabla\Gamma^*(\widetilde{x})=0$, we have by the mean value theorem

$$|\overline{h}(y)-h^{\frac{1}{\alpha-1}}(y)|\leq|\Gamma(\widetilde{y})-\Gamma^*(\widetilde{y})|\leq\|\Gamma\|_{1,\beta-1}|\widetilde{y}-\widetilde{x}|^\beta,\qquad y\in U. \tag{2.5}$$

Let $\rho_\Pi(y)=dist(y,\Pi)$ for $y\in\mathbb{R}^n$ and $D_{\Gamma^*}=\{y\in\mathbb{R}^n:y_n>\Gamma^*(\widetilde{y})\}$. It is clear that $\overline{h}=\sqrt{1+|\nabla\Gamma(\widetilde{x_0})|^2}\rho_\Pi$. So we have by (2.3)

$$\Delta^{\alpha/2}_{D_{\Gamma^*}}\overline{h}^{\alpha-1}(y)=(1+|\nabla\Gamma(\widetilde{x_0})|^2)^{\frac{\alpha-1}{2}}\Delta^{\frac{\alpha}{2}}_{D_{\Gamma^*}}\rho_\Pi^{\alpha-1}(y)=0,\quad y\in D_{\Gamma^*}. \tag{2.6}$$

Denote

$$A=\{y:\Gamma^*(\widetilde{y})<y_n<\Gamma(\widetilde{y}),\ |\widetilde{y}-\widetilde{x}|<1\}\cup\{y:\Gamma(\widetilde{y})<y_n<\Gamma^*(\widetilde{y}),\ |\widetilde{y}-\widetilde{x}|<1\}.$$

Noticing that $\overline{h}^{\alpha-1}(x)=h(x)$ and $B(x,1)\cap D\subset U$ (by the fact that $x_n\leq 1+2^\beta\|\Gamma\|_{1,\beta-1}$), we have by (2.6)

$$\begin{aligned}&\lim_{\varepsilon\downarrow 0}\left|\int_{y\in D,|y-x|>\varepsilon}\frac{h(y)-h(x)}{|x-y|^{n+\alpha}}\,dy\right|\\ &\leq\limsup_{\varepsilon\downarrow 0}\left|\int_{y\in U,|y-x|>\varepsilon}\frac{\overline{h}^{\alpha-1}(y)-\overline{h}^{\alpha-1}(x)}{|x-y|^{n+\alpha}}\,dy\right|+\limsup_{\varepsilon\downarrow 0}\left|\int_{y\in U,|y-x|>\varepsilon}\frac{h(y)-\overline{h}^{\alpha-1}(y)}{|x-y|^{n+\alpha}}\,dy\right|\end{aligned}$$

$$
\begin{aligned}
&+\limsup_{\varepsilon\downarrow 0}\left|\int_{y\in D\setminus U,|y-x|>\varepsilon}\frac{h(y)-h(x)}{|x-y|^{n+\alpha}}\,dy\right|\\
&\le\int_A\frac{|\overline{h}^{\alpha-1}(y)-\overline{h}^{\alpha-1}(x)|}{|x-y|^{n+\alpha}}\,dy+\int_{B(x,1)^c}\frac{|\overline{h}^{\alpha-1}(y)-\overline{h}^{\alpha-1}(x)|}{|x-y|^{n+\alpha}}\,dy\\
&+\int_U\frac{|h(y)-\overline{h}^{\alpha-1}(y)|}{|x-y|^{n+\alpha}}\,dy+\int_{B(x,1)^c}\frac{|h(y)-h(x)|}{|x-y|^{n+\alpha}}\,dy\\
&:=I_1+I_2+I_3+I_4.
\end{aligned}\tag{2.7}
$$

Noticing that $A\subset\{y:|y_n-(x_0)_n|\le 2^{\beta-1}\|\Gamma\|_{1,\beta-1}|\widetilde{y}-\widetilde{x_0}|\ \}$, we have

$$
|x-y|\ge(1+2^{2\beta-2}\|\Gamma\|_{1,\beta-1}^2)^{-1/2}\overline{h}(x)\ge(1+2^{\beta-1}\|\Gamma\|_{1,\beta-1})^{-1}\overline{h}(x),\quad y\in A,
$$

which implies

$$
|x-y|\ge\frac{(1+2^{\beta-1}\|\Gamma\|_{1,\beta-1})^{-1}\overline{h}(x)+|\widetilde{y}-\widetilde{x}|}{2},\quad y\in A.
$$

By (2.5), we also have $\overline{h}(y)\le\|\Gamma\|_{1,\beta-1}|\widetilde{y}-\widetilde{x}|^\beta$ for $y\in A$. Therefore

$$
\begin{aligned}
I_1\le&\int_0^1 dr\int_{|\widetilde{y}-\widetilde{x}|=r}I_A(y)\frac{|\overline{h}^{\alpha-1}(y)-\overline{h}^{\alpha-1}(x)|}{|x-y|^{n+\alpha}}\,m(dy)\\
\le&(\|\Gamma\|_{1,\beta-1}^{\alpha-1}+1)\int_0^{\overline{h}(x)^{\frac{1}{\beta}}\wedge 1}dr\int_{|\widetilde{y}-\widetilde{x}|=r}I_A(y)\frac{\overline{h}(x)^{\alpha-1}}{|x-y|^{n+\alpha}}\,m(dy)\\
&+(\|\Gamma\|_{1,\beta-1}^{\alpha-1}+1)\int_{\overline{h}(x)^{\frac{1}{\beta}}\wedge 1}^1 dr\int_{|\widetilde{y}-\widetilde{x}|=r}I_A(y)\frac{r^{\beta(\alpha-1)}}{|x-y|^{n+\alpha}}\,m(dy)\\
\le&(2\pi)^n(\|\Gamma\|_{1,\beta-1}^{\alpha}+\|\Gamma\|_{1,\beta-1})\int_0^{\overline{h}(x)^{\frac{1}{\beta}}\wedge 1}\overline{h}(x)^{\alpha-1}\Big(\frac{r+(1+2^{\beta-1}\|\Gamma\|_{1,\beta-1})^{-1}\overline{h}(x)}{2}\Big)^{-\alpha+\beta-2}dr\\
&+(2\pi)^n(\|\Gamma\|_{1,\beta-1}^{\alpha}+\|\Gamma\|_{1,\beta-1})\int_{\overline{h}(x)^{\frac{1}{\beta}}\wedge 1}^1 r^{\beta(\alpha-1)}\Big(\frac{r+(1+2^{\beta-1}\|\Gamma\|_{1,\beta-1})^{-1}\overline{h}(x)}{2}\Big)^{-\alpha+\beta-2}dr\\
\le&(2\pi)^n(\|\Gamma\|_{1,\beta-1}^{\alpha}+\|\Gamma\|_{1,\beta-1})2^{\alpha-\beta+2}\frac{(1+2^{\beta-1}\|\Gamma\|_{1,\beta-1})^{\alpha-\beta+1}}{\alpha-\beta+1}\overline{h}(x)^{\beta-2}\\
&+(2\pi)^n(\|\Gamma\|_{1,\beta-1}^{\alpha}+\|\Gamma\|_{1,\beta-1})2^{\alpha-\beta+2}\int_{\overline{h}(x)^{\frac{1}{\beta}}\wedge 1}^1 r^{\alpha\beta-\alpha-2}dr.
\end{aligned}\tag{2.8}
$$

As $\alpha\beta-\alpha-2>\beta^2-2\beta-1$ for $1<\alpha,\beta\le 2$, we get

$$
\int_{\overline{h}(x)^{\frac{1}{\beta}}\wedge 1}^1 r^{\alpha\beta-\alpha-2}dr\le\int_{\overline{h}(x)^{\frac{1}{\beta}}\wedge 1}^1 r^{\beta^2-2\beta-1}dr\le\frac{1}{2\beta-\beta^2}(\overline{h}(x)\wedge 1)^{\beta-2}.\tag{2.9}
$$

The following properties follows from the definitions of $\overline{h}$ and h.

$$
|\overline{h}(x)-\overline{h}(y)|\le(1+\|\Gamma\|_{1,\beta-1})|x-y|,\qquad y\in\mathbb{R}^n,\tag{2.10}
$$

$$
|h(x)^{\frac{1}{\alpha-1}}-h(y)^{\frac{1}{\alpha-1}}|\le(1+2^{\beta-1}\|\Gamma\|_{1,\beta-1})|x-y|,\qquad y\in D,\ |\widetilde{y}|\le 2,\tag{2.11}
$$

$$
\rho(y)^{\alpha-1}\le h(y)\le(1+2^{\beta-1}\|\Gamma\|_{1,\beta-1})^{\alpha-1}\rho(y)^{\alpha-1},\qquad y\in D_1',\ |\widetilde{y}|\le 1.\tag{2.12}
$$

Noticing that $\rho(x)<1$ and $h(y)=0$ for $|\widetilde{y}|>2$, by (2.10)-(2.12)

I_2+I_4

$$\le \int_{B(x,1)^c} \frac{(1+\|\Gamma\|_{1,\beta-1})^{\alpha-1}}{|x-y|^{n+1}}\, dy + \int_{B(x,1)^c} \frac{(1+2^{\beta-1}\|\Gamma\|_{1,\beta-1})^{\alpha-1}}{|x-y|^{n+1}}\, dy + \int_{B(x,1)^c} \frac{h(x)}{|x-y|^{n+\alpha}}\, dy$$
$$\le (2\pi)^n\big((1+\|\Gamma\|_{1,\beta-1})^{\alpha-1} + (1+2^{\beta-1}\|\Gamma\|_{1,\beta-1})^{\alpha-1} + (1+2^{\beta-1}\|\Gamma\|_{1,\beta-1})^{\alpha-1}\big). \tag{2.13}$$

To estimate I_3 we define a transform $\Psi(y) = (\widetilde{z}, z_n)$ by

$$\widetilde{z} = \widetilde{y}, \quad z_n = y_n - \Gamma^*(\widetilde{y}), \quad y \in \mathbb{R}^n.$$

We see that $|\frac{\partial \Psi}{\partial y}| = 1$, where $\frac{\partial \Psi}{\partial y}$ is the Jacobian determinant of Ψ. We can also check that

$$|y_1 - y_2| \ge (1+\|\Gamma\|_{1,\beta-1})^{-1}|\Psi(y_1) - \Psi(y_2)|, \qquad for\ y_1, y_2 \in \mathbb{R}^n,$$

and $\Psi(U) \subset \{y : |\widetilde{y} - \widetilde{x}| < 1,\ |y_n| \le 2^{\beta+1}(\|\Gamma\|_{1,\beta-1} + 1)\}$. Hence by (2.5), the inequality

$$|b^{\alpha-1} - a^{\alpha-1}| \le b^{\alpha-2}|b-a|, \quad b>0, a>0, 1<\alpha<2, \tag{2.14}$$

and applying the transform Ψ, we have

$$\begin{aligned}
I_3 \le& \int_U \frac{(1+\|\Gamma\|_{1,\beta-1})^{n+\alpha+1}|\overline{h}(y)|^{\alpha-2}|\widetilde{y}-\widetilde{x}|^\beta}{|\Psi(x)-\Psi(y)|^{n+\alpha}}\, dy \\
=& \int_{\Psi(U)} \frac{(1+\|\Gamma\|_{1,\beta-1})^{n+\alpha+1}|z_n|^{\alpha-2}|\widetilde{z}-\widetilde{x}|^\beta}{|(\widetilde{x},\overline{h}(x)) - z|^{n+\alpha}}\, dz \\
\le& \int_{-2^{\beta+1}(\|\Gamma\|_{1,\beta-1}+1)}^{2^{\beta+1}(\|\Gamma\|_{1,\beta-1}+1)} dr \int_{z_n=r, |\widetilde{z}-\widetilde{x}|<|r-\overline{h}(x)|} \frac{(1+\|\Gamma\|_{1,\beta-1})^{n+\alpha+1}|r|^{\alpha-2}|\widetilde{z}-\widetilde{x}|^\beta}{|(\widetilde{x},\overline{h}(x)) - z|^{n+\alpha}}\, m(dz) \\
&+ \int_{-2^{\beta+1}(\|\Gamma\|_{1,\beta-1}+1)}^{2^{\beta+1}(\|\Gamma\|_{1,\beta-1}+1)} dr \int_{z_n=r, |r-\overline{h}(x)|\le|\widetilde{z}-\widetilde{x}|\le 1} \frac{(1+\|\Gamma\|_{1,\beta-1})^{n+\alpha+1}|r|^{\alpha-2}}{|\widetilde{z}-\widetilde{x}|^{n+\alpha-\beta}}\, m(dz) \\
\le& \frac{(2\pi)^n}{\beta+1} \int_0^{2^{\beta+1}(\|\Gamma\|_{1,\beta-1}+1)} \frac{(1+\|\Gamma\|_{1,\beta-1})^{n+\alpha+1}|r|^{\alpha-2}}{|\overline{h}(x)-r|^{\alpha-\beta+1}}\, dr \\
&+ \frac{(2\pi)^n}{\alpha-\beta+1} \int_0^{2^{\beta+1}(\|\Gamma\|_{1,\beta-1}+1)} \frac{(1+\|\Gamma\|_{1,\beta-1})^{n+\alpha+1}|r|^{\alpha-2}}{|\overline{h}(x)-r|^{\alpha-\beta+1}}\, dr \\
\le& \frac{2(2\pi)^n}{\alpha-\beta+1}\Bigg(\int_0^{2\overline{h}(x)} \frac{(1+\|\Gamma\|_{1,\beta-1})^{n+\alpha+1}}{r^{2-\alpha}|r-\overline{h}(x)|^{\alpha-\beta+1}}\, dr + \int_{2\overline{h}(x)}^{2^{\beta+1}(\|\Gamma\|_{1,\beta-1}+1)} \frac{(1+\|\Gamma\|_{1,\beta-1})^{n+\alpha+1}}{(r-\overline{h}(x))^{-\beta+3}}\, dr\Bigg) \\
\le& \frac{2(2\pi)^n(1+\|\Gamma\|_{1,\beta-1})^{n+\alpha+1}}{\alpha-\beta+1}\Bigg(\int_0^2 \frac{\overline{h}(x)^{\beta-2}}{r^{2-\alpha}|r-1|^{\alpha-\beta+1}}\, dr + \frac{1}{2-\beta}\overline{h}(x)^{\beta-2}\Bigg).
\end{aligned} \tag{2.15}$$

Combining (2.7)-(2.9),(2.13) and (2.15), we get (2.4). $\square$

Remark 2.1. *Estimates (2.4) may not hold if we take $\beta = \alpha$ in Lemma 2.1. For $n = 2$, $\Gamma(x_1) = |x_1|^\beta$ and $x^* = (0,t)$ with $t > 0$, we can check that $\int_U \frac{h(y)-\overline{h}^{\alpha-1}(y)}{|x^*-y|^{2+\alpha}}\, dy = -\infty$ and I_1, I_2, I_4 are all finite. This gives $\Delta_{D_\Gamma}^{\alpha/2} h(x^*) = -\infty$. When $\alpha < \beta < 2$, we can also prove that $\Delta_{D_\Gamma}^{\frac{\alpha}{2}}\rho^{\alpha-1}(x^*)$ may take $-\infty$. We still consider the above example. Let x_0^* be the point on ∂D_Γ such that $|x_0^* - x^*| = \rho(x)$ and $(x_0^*)_1 > 0$. Let*

$$U = \{(y_1, y_2) : \quad y_2 > |y_1|^\beta \ or\ y_1 \le 0\} \cap \{(y_1,y_2) : y_2 > 0\}$$

and denote the distance function to ∂U by $\xi(x)$. Since ξ is smooth in a neighborhood of x^*, we know that $\Delta_U^{\frac{\alpha}{2}}\xi^{\alpha-1}(x^*)$ is finite. On the other hand,

$$\int_{D_\Gamma} \frac{\rho(y)^{\alpha-1} - \xi(y)^{\alpha-1}}{|x^*-y|^{2+\alpha}} dy = -\infty.$$

Hence we have $\Delta_{D_\Gamma}^{\frac{\alpha}{2}}\rho^{\alpha-1}(x^*) = -\infty$.

Recall $w_p(y) = y_n^p$ for $y \in \mathbb{R}^n_+$. By (5.4) in [12]

$$\Delta^{\alpha/2}_{\mathbb{R}^n_+} w_p(x) = \mathcal{A}(n, -\alpha)\frac{\omega_{n-1}}{2}\mathcal{B}(\frac{\alpha+1}{2}, \frac{n-1}{2})\gamma(\alpha, p)x^{p-\alpha}, \quad x \in \mathbb{R}^n_+, \ \ p \in (-1, \alpha), \qquad (2.16)$$

where ω_{n-1} is the $(n-2)$-dimensional Lebesgue measure of the unit sphere in $\mathbb{R}^{n-1}$, $\mathcal{B}$ is the Beta function and $\gamma(\alpha, p) = \int_0^1 \frac{(t^p-1)(1-t^{\alpha-p-1})}{(1-t)^{1+\alpha}}\,dt$. In what follows we denote the constant on the right hand side of (2.16) by $C(n, \alpha, p)$.

Lemma 2.2. *Let α, Γ and D be described in Lemma 2.1 and let p be a number such that $\alpha > p > \alpha - 1$. Define function $h_p(x) = (x_n - \Gamma(\widetilde{x}))^p I_{\{|\widetilde{x}|<2\}}$ on $D = D_\Gamma$. Then there exists constant $A_2 = A_2(n, \alpha, \beta, p, \|\Gamma\|_{1,\beta-1})$ such that*

$$\Delta^{\alpha/2}_D h_p(x) \geq A_2 \rho(x)^{p-\alpha}, \qquad x \in D'_{1/A_2}, \ |\widetilde{x}| < 1. \qquad (2.17)$$

Proof We use the definitions and the notations in the proof of Lemma 2.1. Following the arguments in (2.7), for $x \in D'_1$ with $|\widetilde{x}| < 1$ we have

$$\begin{aligned}
&\lim_{\varepsilon\downarrow 0}\int_{y\in D, |y-x|>\varepsilon} \frac{h_p(y) - h_p(x)}{|x-y|^{n+\alpha}}\,dy \\
\geq & C(n, \alpha, p)x^{p-\alpha} - \int_A \frac{|\overline{h}^p(y) - \overline{h}^p(x)|}{|x-y|^{n+\alpha}}\,dy - \int_{B(x,1)^c} \frac{|\overline{h}^p(y) - \overline{h}^p(x)|}{|x-y|^{n+\alpha}}\,dy \\
& - \int_U \frac{|h_p(y) - \overline{h}^p(y)|}{|x-y|^{n+\alpha}}\,dy - \int_{B(x,1)^c} \frac{|h_p(y) - h_p(x)|}{|x-y|^{n+\alpha}}\,dy \\
= & C(n, \alpha, p)x^{p-\alpha} - I_1 - I_2 - I_3 - I_4. \qquad (2.18)
\end{aligned}$$

By similar calculations as in Lemma 2.1, we can find constant k_1 such that

$$\begin{aligned}
&I_1 \leq k_1(\rho(x)^{\beta+p-\alpha-1} \vee 1 + |\ln \rho(x)|), \\
&I_3 \leq k_1(\rho(x)^{\beta+p-\alpha-1} \vee 1 + |\ln \rho(x)|), \qquad I_2 + I_4 \leq k_1. \qquad (2.19)
\end{aligned}$$

Noticing that $p - \alpha < 0$, $\beta > 1$ and $C(n, \alpha, p) > 0$, we obtain (2.17) by (2.18) and (2.19). $\square$

Lemma 2.3. *Let α, Γ, D, $h_{\alpha-1}$ and h_p be objects described in Lemmas 2.1 and 2.2. Let f be a bounded function in $C^1(\overline{D})$. Then there exists constant $A_3 = A_3(n, \alpha, p, \sup_{y\in\mathbb{R}^n} |f(y)|, \sup_{|y|<2} |\nabla f(y)|)$ such that for $x \in D \cap B(0, 1)$*

$$\int_D \frac{|(f(y) - f(x))(h_p(y) - h_p(x))|}{|y-x|^{n+\alpha}}\,dy \leq \begin{cases} A_3(|\log \rho(x)| + 1), & p = \alpha - 1, \\ A_3, & \alpha - 1 < p < \alpha. \end{cases} \qquad (2.20)$$

Proof We only prove the lemma for $p = \alpha - 1$ because the others can be proved similarly. Denote $h_{\alpha-1}$ by h and let $x \in D \cap B(0, 1)$. By (2.11),(2.12) and (2.14),

$$\begin{aligned}
&\int_D \frac{|(f(y) - f(x))(h(y) - h(x))|}{|y-x|^{n+\alpha}}\,dy \\
&\leq \sup_{|y|<2} |\nabla f(y)| \int_{D\cap\{\rho(x)<|y-x|\leq 1\}} \frac{|h(y)^{\frac{1}{\alpha-1}} - h(x)^{\frac{1}{\alpha-1}}|^{\alpha-1}}{|y-x|^{n+\alpha-1}}\,dy \\
&+ \sup_{|y|<2} |\nabla f(y)| \int_{D\cap B(x,\rho(x))} \frac{|h(y)^{\frac{1}{\alpha-1}} - h(x)^{\frac{1}{\alpha-1}}| h(x)^{\frac{\alpha-2}{\alpha-1}}}{|y-x|^{n+\alpha-1}}\,dy
\end{aligned}$$

$$
\begin{aligned}
&+2\sup_{y\in\mathbb{R}^n}|f(y)|\int_{D\cap B(x,1)^c\cap\{y:|\widetilde{y}|\le 2\}}\frac{|h(y)^{\frac{1}{\alpha-1}}-h(x)^{\frac{1}{\alpha-1}}|^{\alpha-1}}{|y-x|^{n+\alpha}}\,dy\\
&+2\sup_{y\in\mathbb{R}^n}|f(y)|\int_{\{y:|\widetilde{y}|>2\}}\frac{h(x)}{|y-x|^{n+\alpha}}\,dy\\
&\le\sup_{|y|<2}|\nabla f(y)|\int_{D\cap\{\rho(x)<|y-x|\le 1\}}\frac{(1+2^\beta\|\Gamma_z\|_{1,\beta-1})^{\alpha-1}}{|y-x|^n}\,dy\\
&+\sup_{|y|<2}|\nabla f(y)|\int_{D\cap B(x,\rho(x))}\frac{(1+2^\beta\|\Gamma_z\|_{1,\beta-1})^{\alpha-1}\rho(x)^{\alpha-2}}{|y-x|^{n+\alpha-2}}\,dy\\
&+2\sup_{y\in\mathbb{R}^n}|f(y)|\int_{D\cap B(x,1)^c}\frac{(1+2^\beta\|\Gamma_z\|_{1,\beta-1})^{\alpha-1}}{|y-x|^{n+1}}\,dy+2\sup_{y\in\mathbb{R}^n}|f(y)|\frac{(1+2^\beta\|\Gamma_z\|_{1,\beta-1})^{\alpha-1}}{\alpha}(2\pi)^n\\
&\le(\sup_{|y|<2}|\nabla f(y)|+2\sup_{y\in\mathbb{R}^n}|f(y)|)(1+2^\beta\|\Gamma_z\|_{1,\beta-1})^{\alpha-1}(2\pi)^n(-\ln\rho(x)+\frac{1}{2-\alpha}+1+\frac{1}{\alpha}),
\end{aligned}
$$

which completes the proof. □

For C^1 function κ on $\overline{G}\times\overline{G}$ and $Q\in\partial G$, in the proposition below we denote

$$C_0=\sup_{x,y\in B(Q,1)\cap G}|\nabla_y\kappa(x,y)|. \tag{2.21}$$

Proposition 2.4. *Let $1<\alpha<\beta\le 2$ and G a $C^{1,\beta-1}$ open set in $\mathbb{R}^n$ with characteristics $r_0=1$ and Λ. Let κ be a C^1 function on $\overline{G}\times\overline{G}$ taking values between two positive numbers C_1 and C_2. Then for $\alpha-1\le p<\alpha$ and $Q\in\partial G$, there exist function u_p and positive constants $A_4=A_4(\Lambda)$, $A_5=A_5\big(n,\alpha,\beta,p,\Lambda,C_0,C_1,C_2\big)$ such that*

$$A_4^{-1}I_{G\cap B(Q,2/3)}\rho(x)^p\le u_p(x)\le A_4 I_{G\cap B(Q,2/3)}\rho(x)^p,\quad x\in G, \tag{2.22}$$

and

$$\Delta_G^{\frac{\alpha}{2},\kappa}u_p(x)\ge A_5\rho(x)^{p-\alpha},\qquad x\in G\cap B(Q,1/A_5),\quad \alpha-1<p<\alpha, \tag{2.23}$$

$$|\Delta_G^{\frac{\alpha}{2},\kappa}u_{\alpha-1}(x)|\le\begin{cases} A_5\rho(x)^{\beta-2}, & x\in G\cap B(Q,1/2),\quad if\ \beta<2,\\ A_5|\ln\rho(x)|, & x\in G\cap B(Q,1/2),\quad if\ \beta=2.\end{cases} \tag{2.24}$$

Proof Without loss of generality, we assume that $Q=0$ and take the coordinate system CS_Q (see (1.3)). Define functions $u_p(x)=(x_n-\Gamma_Q(\widetilde{x}))^p I_{G\cap\{B(Q,2/3)\}}$ on G and $v_p(x)=(x_n-\Gamma_Q(\widetilde{x}))^p I_{|\widetilde{x}|<2}$ on D_{Γ_Q} for $\alpha-1\le p<\alpha$. It is easy to see that (2.22) holds. When $\kappa\equiv 1$, noticing that for $x\in G\cap B(0,1/2)$ the integral in (1.1) for u_p on $G\cap B(Q,2/3)^c$ and v_p on $D_{\Gamma_Q}\cap B(Q,2/3)^c$ can be bounded by constants depending on n and α, we can prove this proposition by Lemma 2.1 and Lemma 2.2. For general cases, the conclusion can be proved by the case $\kappa\equiv 1$, Lemma 2.3 and the following identity:

$$
\begin{aligned}
&\Delta_G^{\frac{\alpha}{2},\kappa}h(x)\\
=&\mathcal{A}(n,-\alpha)\lim_{\varepsilon\downarrow 0}\int_{y\in G,|y-x|>\varepsilon}\frac{(\kappa(x,y)-\kappa(x,x))(h(y)-h(x))}{|x-y|^{n+\alpha}}dy+\kappa(x,x)\Delta_G^{\alpha/2}h(x).
\end{aligned}\tag{2.25}
$$

□

3 Harnack inequalities of $\Delta_G^{\frac{\alpha}{2},\kappa}$

The following example can be found in [23].

Example 3.1. *Let* $G = \mathbb{R}^n_+$ *and* $\overline{y} = (\widetilde{y}, -y_n)$ *for* $y = (\widetilde{y}, y_n)$*. For* $\kappa(x,y) = 1 + \frac{|x-y|^{n+\alpha}}{|x-\overline{y}|^{n+\alpha}}$, $\Delta_G^{\frac{\alpha}{2},\kappa}$ *is the formal generator of the subordinate reflected Brownian motion on* $\overline{\mathbb{R}^n_+}$*. When* $G = (0,1)$ *and* $\kappa(x,y) = \sum_{k=-\infty}^{\infty} |x-y|^{1+\alpha}/|x \pm y + 2k|^{1+\alpha}$, $\Delta_G^{\frac{\alpha}{2},\kappa}$ *is the formal generator of the subordinate reflected Brownian motion on* $[0,1]$.

Remark 3.1. *Define function* $w_p(y) = y_n^p$ *for* $y \in \mathbb{R}^n_+$*. When* $\kappa(x,y) = \frac{|x-y|^{n+\alpha}}{|x-\overline{y}|^{n+\alpha}}$*, we have (see[23])*

$$\Delta_{\mathbb{R}^n_+}^{\frac{\alpha}{2},\kappa} w_p(x) = \mathcal{A}(n,-\alpha)\frac{\omega_{n-1}}{2}\mathcal{B}(\frac{\alpha+1}{2}, \frac{n-1}{2})\overline{\gamma}(\alpha,p)x^{p-\alpha}, \quad x \in \mathbb{R}^n_+, \quad p \in (-1,\alpha), \tag{3.1}$$

where $\overline{\gamma}(\alpha,p) = \int_0^1 (t^p - 1)(1 - t^{\alpha-p-1})/(1+t)^{1+\alpha}\, dt$*. This gives the same (super,sub) harmonic functions as the homogeneous case in (2.16), which will be used later.*

Notice that the derivatives of κ in the examples above are not bounded. To give results including these examples we introduce the following condition. Let $0 < C_1 < C_2$, $C_3 > 0$ and $\gamma \le 0$. We say that κ or the reflected stable-like process (X_t) satisfies condition $[C_1, C_2, C_3, \gamma]$ if

$$C_1 < \kappa(x,y) < C_2, \quad x,y \in \overline{G}; \qquad |\kappa(x,y) - \kappa(x,x)| < C_3(\rho(x)^\gamma \vee 1)|x-y|, \quad x,y \in G. \tag{3.2}$$

We can check that functions κ in the Example 3.1 above satisfy condition $[C_1, C_2, C_3, -1]$ for some constants $C_1, C_2, C_3 > 0$.

Next we prepare a stochastic calculus formula for (X_t). For a measurable function f on G, denote $f \in \mathcal{L}^1_u(G)$ if

$$\sup_{x \in G} \int_G \frac{|f(x) - f(y)|}{(1+|x-y|)^{n+\alpha}}\, dy < \infty. \tag{3.3}$$

For any subset $U \subseteq \mathbb{R}^n$ and $0 < \gamma \le 1$, we say that u is uniformly γ-Hölder continuous on U if

$$\sup_{(y,z) \in U \times U} \frac{|u(y) - u(z)|}{|z-y|^\gamma} < \infty. \tag{3.4}$$

We shall denote ($u \in C^{1,\gamma}(U)$) $u \in C^\gamma(U)$ if (all the first derivatives of u) u is uniformly γ-Hölder continuous on U. For any $\delta > 0$ and $A \subseteq \mathbb{R}^n$, define $\tau_A = \inf\{t > 0 : X_t \in A^c\}$ and $A^\delta = \{y : |y - x| < \delta,\ for\ some\ x \in A\}$. For any relatively open subset A of $\overline{G}$, we denote by (p_t^A) and G^A the probability transition function and the Green function of (X_t) killed upon leaving A, respectively. In [23], a semi-martingale decomposition of $f(X_t)$ is given for $f \in C_c^2(\overline{G})$ (see [24] for the homogeneous case). To consider more general functions, we prove the following results.

Proposition 3.1. *Let* G *be a Lipschitz open set in* $\mathbb{R}^n$*. For* $1 \le \alpha < \beta \le 2$*, let* κ *be a symmetric function on* $\overline{G} \times \overline{G}$ *satisfying condition* $[C_1, C_2, C_3, \gamma]$ *with* $0 \ge \gamma > \alpha - 3$*. For* $0 < \alpha < \beta \le 1$*, let* κ *be a measurable symmetric function on* $\overline{G} \times \overline{G}$ *bounded between* C_1 *and* C_2*. Then for* f *belonging to*

$$C^{1,\beta-1}(\overline{G}) \cap \mathcal{L}^1_u(G), \quad 1 \le \alpha < \beta \le 2; \qquad C^\beta(\overline{G}) \cap \mathcal{L}^1_u(G), \quad 0 < \alpha < \beta \le 1, \tag{3.5}$$

we have

$$f(X_t) = f(x_0) + M_t + \int_0^t \Delta_G^{\frac{\alpha}{2},\kappa} f(X_s)\, ds, \quad a.s.\ \ x_0 \in \overline{G}, \tag{3.6}$$

where $(M_t)_{t\geq 0}$ is a martingale. If A is a relatively open set in $\overline{G}$ and, for some $\delta>0$, f satisfies (3.5) with $\overline{G}$ replaced by $\overline{G}\cap A^\delta$, then

$$E_{x_0}(f(X_{t\wedge\tau_A}))=f(x_0)+E_{x_0}(\int_0^{t\wedge\tau_A}\Delta_G^{\frac{\alpha}{2},\kappa}f(X_s)ds),\qquad x_0\in A,\ t\geq 0. \tag{3.7}$$

Moreover, if $P_{x_0}(X_{\tau_A}\in\partial A)=0$ and f is a positive function such that $f=0$ on A, then (3.7) still holds.

Proof Assume that f satisfies (3.5). For $1\leq\alpha<\beta\leq 2$, by (3.4), the derivatives of f at point $x\in G$ can be bounded by $k_1(1+|x|)$ for some constant $k_1>0$. By this estimate, (3.3) and straightforward calculations for the integral in (1.1) on sets $B(x,\rho(x))$, $(G\cap B(x,1))\setminus B(x,\rho(x))$ and $G\setminus B(x,1)$ respectively, we can prove that for some constant k_2,

$$|\Delta_G^{\frac{\alpha}{2},\kappa}f(x)|\leq k_2(1+|x|)\rho(x)^{(2+\gamma-\alpha)\wedge(1-\alpha)},\qquad x\in G_1',\ \ 1\leq\alpha<\beta\leq 2. \tag{3.8}$$

By (3.3) and (3.4) we can find constant k_3 such that

$$|\Delta_G^{\frac{\alpha}{2},\kappa}f(x)|\leq k_3,\qquad x\in G_1,\quad 1\leq\alpha<\beta\leq 2. \tag{3.9}$$

By calculating the integral in (1.1) on sets $G\cap B(x,1)$ and $G\setminus B(x,1)$ respectively, we can also check

$$|\Delta_G^{\frac{\alpha}{2},\kappa}f(x)|\leq k_4,\qquad x\in G,\ \ 0<\alpha<\beta\leq 1, \tag{3.10}$$

for some constant k_4. Noticing that $(2+\gamma-\alpha)\wedge(1-\alpha)>-1$, with the help of the heat kernel estimates in [17] and (3.8)-(3.10) we can prove that $E_x(\int_0^1|\Delta_G^{\frac{\alpha}{2},\kappa}f(X_t)|dt)$ is a bounded function on $\overline{G}$ (c.f. Lemma 4.6 [24]). This implies that $E_x(\int_0^t|\Delta_G^{\frac{\alpha}{2},\kappa}f(X_t)|dt)$ is a bounded function on $\overline{G}$ for any $t>0$. Thus we can prove (3.6) by Theorem 5.25 [21] at time $t\wedge\tau_{B(0,n)}$ (c.f. Theorem 4.1 [24]) and letting $n\to\infty$. Formula (3.7) is a consequence of (3.6) by approximation procedure. □

For a relatively open set A in $\overline{G}$, we say that A has outer cone property in $\overline{G}$ if, for some $\eta>0$ and each $Q\in\partial A$, there is a cone in $\overline{G}\setminus A$ isometric to $\{x\in\mathbb{R}^n:|(x_1,\cdots,x_{n-1})|<\eta|x_n|\}$ and taking Q as the vertex.

Proposition 3.2. *Let α, G and κ be the same as in Proposition 3.1. Let $A\subseteq\overline{G}$ be a relatively open set with outer cone property in $\overline{G}$ and define $\tau=\inf\{t>0:X_t\in A^c\}$. Then the distribution of X_τ is absolutely continuous on $\overline{G}\setminus A$ when (X_t) starting from A. For any $t>0$, we have*

$$\begin{aligned}&P_x\{X_\tau I_{\{\tau\leq t\}}\in dy\}/dy\\ =&\mathcal{A}(n,-\alpha)\int_0^t\Big(\int_A\frac{\kappa(z,y)p^A(s,x,z)}{|z-y|^{n+\alpha}}dz\Big)ds,\quad (x,y)\in A\times(\overline{G}\setminus A).\end{aligned} \tag{3.11}$$

Furthermore,

$$\begin{aligned}&P_x\{X_\tau\in dy\}/dy\\ =&\mathcal{A}(n,-\alpha)\int_A\frac{\kappa(z,y)G^A(x,z)}{|z-y|^{n+\alpha}}dz,\quad (x,y)\in A\times(\overline{G}\setminus A).\end{aligned} \tag{3.12}$$

Proof To show that $P_x\{X_\tau\in\partial A\}=0$ for $x\in A$, by the method in Lemma 6 [8], we only need to prove that there exists a constant c such that $P_x\{\tau_{B(x,\rho(x))}\in G\setminus A\}>c$ for any $x\in A$ (the boundedness assumption in [8] is not necessary because A can be approximated by $A\cap B(0,n)$ by letting $n\uparrow\infty$). We omit the proof of this estimate because it is similar to (3.25) below. Thus we can prove (3.11) by Proposition 3.1. Formula (3.12) is a consequence of (3.11). □

Lemma 3.3. *Let $0 < \alpha < 2$ and let G be a Lipschitz open set in $\mathbb{R}^n$. Assume that κ is a symmetric function on $\overline{G} \times \overline{G}$ satisfying condition $[C_1, C_2, C_3, -1]$. Let $\lambda \geq 1$ and define process $((Z_t)_{t\geq 0}, Q_{x_0}) = ((\lambda X_{\lambda^{-\alpha}t})_{t\geq 0}, P_{x_0/\lambda})$ for $x_0 \in \overline{\lambda G}$. Then (Z_t) is a reflected stable-like process on $\overline{\lambda G}$ satisfying condition $[C_1, C_2, C_3, -1]$.*

Proof The conclusion can be proved by checking that the jumping measure of (Z_t) is

$$\frac{\kappa(x/\lambda, y/\lambda)}{|x-y|^{n+\alpha}}\, dxdy, \quad x, y \in \lambda G. \qquad \Box$$

Lemma 3.4. *Let $0 < \alpha < 2$ and let G be a Lipschitz open set in $\mathbb{R}^n$ with characteristics $r_0 = 1$ and Λ. Assume that κ is a symmetric function on $\overline{G} \times \overline{G}$ satisfying condition $[C_1, C_2, C_3, -1]$. Then for $0 < \varepsilon < 1$, there exists constants $A_6 = A_6(n, \alpha, C_2, C_3, \varepsilon)$ and $A_6' = A_6'(n, \alpha, C_1)$ such that for any $0 < r \leq r_0/2$*

$$A_6 r^\alpha \leq \inf_{y \in B(x,(1-\varepsilon)r)} E_y \tau_{B(x,r)} \leq \sup_{y\in\overline{G}} E_y \tau_{B(x,r)} \leq A_6' r^\alpha, \quad x \in G \text{ with } \rho(x) > 2r. \tag{3.13}$$

Moveover, the last inequality in (3.13) holds for all $x \in \overline{G}$ provided $r < r_0/4$, where A_6' depends further on Λ.

Proof By the scaling property in Lemma 3.3 and the Lipschitz condition of G, we can assume that $r = 1$. Choose $f_1 \in C^2(\overline{G})$ such that $0 \leq f_1 \leq 1$ and

$$f_1(y) = 0, \quad y \in B(x, 1-\varepsilon); \qquad f_1(y) = 1, \quad y \in B(x, 1-\varepsilon/2)^c.$$

By direct calculation, we can find a constant $k_1 = k_1(n, \alpha, C_2, C_3, \varepsilon)$ such that $|\Delta_G^{\frac{\alpha}{2},\kappa} f_1(y)| < k_1$ for $y \in B(x, 1)$. Thus we can prove the first inequality in (3.13) by applying formula (3.7) to $E_y(f_1(X_{\tau_{B(x,1)}}))$. Similarly, with the help of Proposition 3.2, the last inequality in (3.13) can be proved by considering function $f_2 = I_{\overline{G}\setminus B(x,1)}$, where we can check that $\Delta_G^{\frac{\alpha}{2},\kappa} f_2(y) > k_2$, $y \in \overline{G} \cap B(x, 1)$, for some constant $k_2 = k_2(n, \alpha, C_1)$. For the last conclusion, $\overline{G} \cap B(x, 1)$ may not have the outer cone property in $\overline{G}$, where we need to replace $B(x, 1)$ by a bigger set in $B(x, 2)$ satisfying this property. $\Box$

The next theorem extends the Harnack inequality for the censored stable process in [12].

Theorem 3.5. *Let $0 < \alpha < 2$ and let G be a Lipschitz open set in $\mathbb{R}^n$ with characteristics r_0 and Λ. Assume that κ is a symmetric function on $\overline{G} \times \overline{G}$ satisfying condition $[C_1, C_2, C_3, -1]$. Let $0 < r \leq 1$, $k \in \{1, 2, \ldots\}$ and $x_1, x_2 \in G$ such that $B(x_1, r) \cup B(x_2, r) \subset G$ and $|x_1 - x_2| < 2^k r$. If $u \geq 0$ is harmonic for (X_t) on $B(x_1, r) \cup B(x_2, r)$, then there exists constant $A_7 = A_7(n, \alpha, C_1, C_2, C_3)$ such that*

$$A_7^{-1} 2^{-k(n+\alpha)} u(x_2) \leq u(x_1) \leq A_7 2^{k(n+\alpha)} u(x_2). \tag{3.14}$$

Proof For simplicity we assume $\kappa \equiv 1$. Let $y \in G$ with $\rho(y) \geq r$. First we prove that there exists a constant $k_1 = k_1(n, \alpha)$ such that

$$u(y_1) \leq k_1 u(y_2), \quad y_1, y_2 \in B(y, r/2), \tag{3.15}$$

provided $u \geq 0$ is harmonic for (X_t) on $B(y, r)$. To show this we only need to prove that

$$u(y_1) \leq k_1 u(y_2), \quad y_1, y_2 \in B(y, r/2) \text{ and } |y_1 - y_2| > r/3. \tag{3.16}$$

Approximating by functions $u_k := E_x\big((u \wedge k)(X_{\tau_{B(y,r)}})\big)$, we can assume that u is bounded. By scaling we can also assume $r = 1$.

Let $y_1, y_2 \in B(y, 1/2)$ such that $|y_1 - y_2| > 1/3$. Suppose that $u(y_1) > Mu(y_2)$ for some big number M and we can construct a sequence of points $(x_k) \in B(y_1, 1/6)$ such that $x_0 = x_1 = y_1$ and

$$|u(x_k)| \geq (1+\delta)^{k-1} Mu(y_2), \quad |x_k - x_{k-1}| \leq 12^{-1}(k-1)^{-2}, \quad k \geq 1, \tag{3.17}$$

for some $\delta > 0$, then the contradiction between the boundedness of u and $\lim_{k\to\infty} u(x_k) = \infty$ leads to (3.16) (here we assume that $u(y_2) > 0$ because $u(y_2) = 0$ implies that $u \equiv 0$ on $G \cap B(y, 1)$, c.f. (3.19) below).

Suppose that (3.17) holds for $k = 1$ and some δ, M which will be fixed later. Setting $B_k = B(x_k, 24^{-1}k^{-2})$ and $\tau_k = \tau_{B_k}$ for $k \geq 1$, we have by Proposition 3.2 and Lemma 3.4

$$\begin{aligned}
&P_{x_k}\{X_{\tau_k} \in G \setminus (2B_k)\} \\
=&\mathcal{A}(n,-\alpha) \int_{B_k} \int_{y \in G\setminus(2B_k)} \frac{G^{B_k}(x_k, z)}{|z-y|^{n+\alpha}} dz dy \\
\geq& 2^{-(n+\alpha)} \mathcal{A}(n,-\alpha) \int_{B_k} G^{B_k}(x_k, z) \int_{y \in G\setminus(2B_k)} \frac{1}{|x_k - y|^{n+\alpha}} dz dy \\
=& 2^{-(n+\alpha)} \mathcal{A}(n,-\alpha)(E_{x_k}\tau_k) \int_{y \in G\setminus(2B_k)} \frac{1}{|x_k - y|^{n+\alpha}} dy \\
\geq& k_1
\end{aligned} \tag{3.18}$$

for some constant $k_1 = k_1(n, \alpha)$. Similarly, by setting $B_0 = B(y_2, 1/6)$ and $\tau_0 = \tau_{B_0}$, we also have

$$u(y_2) = E_{y_2}(u(X_{\tau_0}) I_{X_{\tau_0} \in G\setminus B_0}) \geq k_2(n, \alpha) \int_{y \in G\setminus B_0} \frac{u(y)}{|y_2 - y|^{n+\alpha}} dy. \tag{3.19}$$

By (2.1) and an estimate of $P_{y_2}(\tau_0 \in B(y, 1/2) \setminus (2B_0))$ similar to (3.18), we can find $y_3 \in B(y, 1/2) \setminus (2B_0)$ such that $u(y_3) \leq k_3 u(y_2)$ for some constant $k_3 = k_3(n, \alpha)$. Similar to (3.19), we have

$$u(y_3) \geq k_4(n, \alpha) \int_{y \in B_0} \frac{u(y)}{|y_3 - y|^{n+\alpha}} dy. \tag{3.20}$$

Noticing that $|y - x_k| \geq \frac{1}{12k^2}(|y - y_2| \vee |y - y_3|)$ for $y \in G \setminus (2B_k)$, we have by Proposition 3.2 and Lemma 3.4

$$\begin{aligned}
&E_{x_k}(u(X_{\tau_k}) I_{X_{\tau_k} \in G\setminus(2B_k)}) \\
\leq& 2^{n+\alpha} \mathcal{A}(n,-\alpha)(E_{x_k}\tau_k) \int_{y \in G\setminus(2B_k)} \frac{u(y)}{|x_k - y|^{n+\alpha}} dy \\
\leq& 2^{n+\alpha} (\frac{1}{24k^2})^{\alpha} A_6' \mathcal{A}(n,-\alpha) \int_{y \in G\setminus(2B_k)} \frac{u(y)}{|x_k - y|^{n+\alpha}} dy \\
\leq& k_5(n, \alpha) k^{2n} (\int_{y \in G\setminus B_0} \frac{u(y)}{|y_2 - y|^{n+\alpha}} dy + \int_{y \in B_0} \frac{u(y)}{|y_3 - y|^{n+\alpha}} dy).
\end{aligned} \tag{3.21}$$

By (3.19)-(3.21) and $u(y_3) \leq k_3 u(y_2)$, we have $E_{x_k}(u(X_{\tau_k}) I_{X_{\tau_k} \in G\setminus(2B_k)}) \leq k_6(n, \alpha) k^{2n} u(y_2)$. Thus by (2.1), (3.17) and (3.18) we have

$$(1+\delta)^{k-1} Mu(y_2) \leq u(x_k) \leq (1 - k_1) \sup_{y \in (2B_k)\setminus B_k} u(y) + k_6 k^{2n} u(y_2). \tag{3.22}$$

Now choose $\delta = k_1/2$ and $K_0 = K_0(n,\alpha) \geq 1$ such that for any $M > 1$

$$(1+\delta)^{m-1}M - k_6 m^{2n} \geq \frac{1-k_1}{1-k_1/2}(1+\delta)^{m-1}M, \qquad m \geq K_0. \tag{3.23}$$

If x_k with $k > K_0$ satisfies (3.22) for some $M > 1$, then (3.22) and (3.23) show that there exists x_{k+1} satisfying (3.17) for $k+1$. By (3.22), we can also choose $M = M(n,\alpha) > 1$ big enough such that (3.23) holds for $1 \leq k \leq K_0$. Therefore, we can finish the proof of (3.15) by induction.

Next we assume that $2^k r > |x_1 - x_2| > r$. We have for $x \in B(x_1, r/3)$

$$\begin{aligned}\Delta_G^{\frac{\alpha}{2},\kappa} I_{B(x_2,r/3)}(x) =& \mathcal{A}(n,-\alpha)\int_{y\in B(x_2,r/3)} \frac{1}{|x-y|^{n+\alpha}}\, dy \\ \geq& k_7(n,\alpha) r^{-\alpha} 2^{-k(n+\alpha)}.\end{aligned} \tag{3.24}$$

By (3.7),(3.13) and (3.24) we have

$$\begin{aligned}&P_{x_1}\{X_{\tau_{B(x_1,r/3)}} \in B(x_2, r/3)\} \\ =& E_{x_1}\Big(\int_0^{\tau_{B(x_1,r/3)}} \Delta_G^{\frac{\alpha}{2},\kappa} I_{B(x_2,r/3)}(X_t)dt\Big) \\ \geq& k_7 A_6 3^{-\alpha} 2^{-k(n+\alpha)}.\end{aligned} \tag{3.25}$$

By (3.15) and (3.25),

$$u(x_1) = E_{x_1}(u(X_{\tau_{B(x_1,r/3)}})) \geq k_8(n,\alpha) 2^{-k(n+\alpha)} u(x_2), \tag{3.26}$$

which completes the proof. □

Corollary 3.6. *Let α, G and κ be the same as in Theorem 3.5 and let u be a (X_t) harmonic function in an open subset D of G. Then u is continuous on D.*

Proof Let $x \in D$ and $\rho_D(x) = \inf\{|x-y| : y \in \partial D\}$. By Theorem 3.5 we see that u is bounded on $B(x, 2\rho_D(x)/3)$. Set $\tau = \tau_{B(x,\rho_D(x)/3)}$. By the strong Markov property, we have

$$u(y) = E_y u(X_\tau) = E_y[u(x_\tau) I_{t\geq\tau}] + E_y[u(X_t) I_{t<\tau}]. \tag{3.27}$$

By the continuity of the heat kernel in [17], we see $E.[u(X_t)I_{\{t<\tau\}}] \in C(B(x, \rho_D(x)/3))$(c.f. Proposition 3.6 [25]). On the other hand,

$$\begin{aligned}&\Big|E_y[u(X_\tau)I_{\{t\geq\tau\}}]\Big| \\ \leq& \Big(\sup_{z\in(B(x,2\rho_D(x)/3))} |u(z)|\Big) P_y\{t \geq \tau, X_\tau \in (B(x, 2\rho_D(x)/3))\} + \Big|E_y[u(X_\tau) I_{t\geq\tau} I_{X_\tau \notin B(x,2\rho_D(x)/3)}]\Big|.\end{aligned}$$

Therefore, by (3.11), (3.27), Theorem 3.5 and the dominated convergence theorem, we need only to check that $P_y\{t \geq \tau\}$ converges to zero uniformly on $B(x, \rho_D(x)/3)$ when $t \downarrow 0$. This follows from facts that $P_y\{t \geq \tau\} = 1 - P_y\{t < \tau\} \in C_b(B(x, \rho_D(x)/3))$ and $\lim_{t\downarrow 0} P_y\{t \geq \tau\} = 0$ for $y \in B(x, \rho_D(x)/3)$. □

4 Boundary Harnack inequality of $\Delta_G^{\frac{\alpha}{2},\kappa}$ on $C^{1,\beta-1}$ ($C^{1,1}$) open sets

Next we assume that $0 \in \partial G$ and choose the coordinate system CS_0. For $x \in \mathbb{R}^n$, $r > 0$, let $\triangle(x, a, r)$ be the box defined by

$$\triangle(x, a, r) = \{y = (\widetilde{y}, y_n) \in G : 0 < y_n - \Gamma_0(\widetilde{y}) < a, |\widetilde{y} - \widetilde{x}| < r\}. \tag{4.1}$$

The following result is a special case of Theorem 1.1.

Lemma 4.1. *Let $1 < \alpha < \beta \le 2$ and let G be a $C^{1,\beta-1}$ open set with characteristics $r_0 = 1$ and Λ. Assume that κ satisfies the conditions in Proposition 2.4. Then there exist constants $A_8 = A_8(n, \alpha, \beta, \Lambda, C_0, C_1, C_2) < 1/2$ and $A_9 = A_9(n, \alpha, \beta, \Lambda, C_0, C_1, C_2)$ such that*

$$\begin{aligned} A_9^{-1}\rho(x)^{\alpha-1} \le & P_x\{X_{\tau_{\triangle(0,A_8,A_8)}} \in \triangle(0, 2A_8, A_8)\} \\ \le & P_x\{X_{\tau_{\triangle(0,A_8,A_8)}} \in G\} \ \le A_9\rho(x)^{\alpha-1} \end{aligned} \tag{4.2}$$

for $x \in \triangle(0, A_8, A_8)$ with $\widetilde{x} = 0$.

Proof We assume that $\kappa \equiv 1$ because the proof is the same for the general case. Let $p = (\alpha - 1 + ((\alpha + \beta - 2) \wedge 1))/2$ and define

$$v_1(y) = u_{\alpha-1}(y) + u_p(y),$$

where $u_{\alpha-1}$ and u_p are functions defined in Proposition 2.4. Since $p - \alpha > \beta - 2$, by Proposition 2.4, there exists $k_1 = k_1(n, \alpha, \beta, \Lambda)$ such that $\triangle(0, 2k_1, 2k_1) \subseteq B(0, r_0)$ and

$$\Delta_G^{\alpha/2} v_1(y) \ge 0, \qquad y \in \triangle(0, k_1, k_1). \tag{4.3}$$

Let ϕ be a C^2 function on $\overline{G}$ such that

$$\phi(y) = |\widetilde{y}| = y_1^2 + \ldots + y_{n-1}^2, \quad |y| < 1; \qquad 1 \le \phi(y) \le 2, \quad |y| \ge 1.$$

Define

$$v_2(y) = u_{\alpha-1}(y) - u_p(y)/(2A_4^2) + 12k_1^{-2}A_4^3\phi(y).$$

By Lemma 3.4 [12], we have $|\Delta_G^{\alpha/2}\phi(y)| \le k_2(\rho(y)^{1-\alpha} \vee 1)$, $y \in \triangle(0, k_1, k_1)$, for some constant $k_2 = k_2(n, \alpha)$. Thus by $p - \alpha < 1 - \alpha$ and Proposition 2.4, there exists $m = m(n, \alpha, \beta, \Lambda) \le k_1/2$ such that

$$\Delta_G^{\alpha/2} v_2(y) \le 0, \qquad y \in \triangle(0, m, k_1). \tag{4.4}$$

Since $v_2 \ge 3A_4^3$ on $G \setminus \triangle(0, \infty, k_1/2)$ and $v_2(y) \le A_4\rho(y)^{\alpha-1}$ for $y \in G \cap B(0, r_0)$ with $\widetilde{y} = 0$, we have by applying (3.7) and (4.11)

$$P_x\{X_{\tau_{\triangle(0,m,k_1/2)}} \in G \setminus \triangle(0, \infty, k_1/2)\} \le 3^{-1}A_4^{-2}\rho(x)^{\alpha-1} \tag{4.5}$$

for $x \in \triangle(0, m, k_1)$ *with* $\widetilde{x} = 0$.

Noticing that $\sup_{y\in G} |v_1(y)| \le 2A_4$ and $v_1(y) \ge A_4^{-1}\rho(y)^{\alpha-1}$ for $y \in G \cap B(0, r_0)$ with $\widetilde{y} = 0$, by (3.7) and (4.3), we have

$$P_x\{X_{\tau_{\triangle(0,m,k_1)}} \in G \setminus \triangle(0, m, k_1/2)\} \ \ge 2^{-1}A_4^{-2}\rho(x)^{\alpha-1}, \quad x \in \triangle(0, m, k_1) \text{ and } \widetilde{x} = 0. \tag{4.6}$$

Combing (4.5) and (4.6), we have

$$P_x\{X_{\tau_{\triangle(0,m,k_1/2)}} \in \triangle(0, \infty, k_1/2) \setminus \triangle(0, m, k_1/2)\} \ge 6^{-1}A_4^{-2}\rho(x)^{\alpha-1} \tag{4.7}$$

for $x \in \triangle(0, m, k_1)$ *with* $\widetilde{x} = 0$. By (4.7) and (3.12), we can find a constant $k_2 = k_2(k_1, m, \Lambda)$

$$P_x\{X_{\tau_{\triangle(0,m,k_1/2)}} \in \triangle(0, k_1, k_1/2) \setminus \triangle(0, m, k_1/2)\} \geq k_2 A_4^{-2} \rho(x)^{\alpha-1}. \tag{4.8}$$

By (4.8), for $x \in \triangle(0, m, k_1)$ with $\widetilde{x} = 0$

$$\begin{aligned}
&P_x\{X_{\tau_{\triangle(0,k_1,k_1)}} \in \triangle(0, 2k_1, k_1) \setminus \triangle(0, k_1, k_1)\} \\
\geq &P_x\{X_{\tau_{\triangle(0,m,k_1/2)}} \in \triangle(0, 2k_1, k_1) \setminus \triangle(0, k_1, k_1)\} \\
\geq &P_x\{X_{\tau_{\triangle(0,m,k_1/2)}} \in \triangle(0, k_1, k_1/2) \setminus \triangle(0, m, k_1/2)\} \cdot \\
&\sup_{y \in \triangle(0,k_1,k_1/2) \setminus \triangle(0,m,k_1/2)} P_y\{X_{\tau_{\triangle(0,k_1,k_1)}} \in \triangle(0, 2k_1, k_1) \setminus \triangle(0, k_1, k_1)\} \\
\geq &k_2 k_3 A_4^{-2} \rho(x)^{\alpha-1},
\end{aligned} \tag{4.9}$$

where we use the fact that for some $k_3 = k_3(k_1, m, \Lambda)$

$$\begin{aligned}
&P_y\{X_{\tau_{\triangle(0,k_1,k_1)}} \in \triangle(0, 2k_1, k_1) \setminus \triangle(0, k_1, k_1)\} \geq k_3, \\
&\quad y \in \triangle(0, k_1, k_1/2) \setminus \triangle(0, m, k_1/2).
\end{aligned} \tag{4.10}$$

which can be proved by the same calculation as (3.25). Setting $A_8 = k_1$, (4.5), (4.9) and (4.10) yield the first inequality of (4.2) for $x \in \triangle(0, k_1, k_1)$ *with* $\widetilde{x} = 0$.

Set $v_3(x) = v_2(x) I_{x \in G, |x| < 1/2} + I_{x \in G, |x| \geq 1/2}$, by Proposition 2.4 we can choose k_1 small enough such that

$$\Delta_G^{\alpha/2} v_3(y) \leq 0, \qquad y \in \triangle(0, k_1, k_1). \tag{4.11}$$

This estimate and Proposition 3.1 gives the second inequality of (4.2). $\square$

Lemma 4.2. (*Carleson estimate*) *Let $1 < \alpha < \beta \leq 2$ and let G be a $C^{1,\beta-1}$ open set with characteristics $r_0 = 1$ and Λ. Assume that κ satisfies the conditions in Proposition 2.4. Let $Q = 0 \in \partial G$ and assume that $u \geq 0$ is a function on G which is not identical to zero, harmonic on $G \cap B(Q, 1)$ and vanishes on $\partial G \cap B(Q, 1)$ continuously. Then there exists a constant $A_{10} = A_{10}(n, \alpha, \beta, \Lambda, C_0, C_1, C_2)$ such that*

$$u(x) \leq A_{10} u(x_0), \qquad x \in G \cap B(Q, 1/2), \tag{4.12}$$

where $x_0 = (0, 1/2)$ in the coordinate system CS_Q.

Proof By chain arguments, we only need to prove (4.12) for $x \in G \cap B(Q, 1/8)$. By multiplying a constant we can also assume that $u(x_0) = 1$. Choose $0 < \gamma < \alpha/(n + \alpha)$ and define

$$B_0 = G \cap B(x, 2\rho(x)), \qquad B_1 = B(x, \rho(x)^\gamma).$$

Set

$$B_2 = B(x_0, \rho(x_0)/3), \qquad B_3 = B(x_0, 2\rho(x_0)/3)$$

and

$$\tau_1 = \inf\{t > 0 : X_t \notin B_0\}, \qquad \tau_2 = \inf\{t > 0 : X_t \notin B_2\}.$$

By (4.2) and scaling, we can find a constant $\delta = \delta(n, \alpha, \beta, \Lambda, C_0, C_1, C_2)$ such that

$$P_x(X_{\tau_1} \in \partial G) > \delta, \qquad x \in G \cap B(Q, 1/4). \tag{4.13}$$

By Harnack inequality (3.14), there exists $\beta' = \beta(n, \alpha, \beta, C_0, C_1, C_2)$ such that

$$u(x) < \rho(x)^{-\beta'} u(x_0), \quad x \in G \cap B(Q, 1/4). \tag{4.14}$$

Since u is harmonic on $G\cap B(Q,1)$, we have for $x\in G\cap B(Q,1/4)$

$$u(x) = E_x(u(X_{\tau_1})I_{X_{\tau_1}\in B_1}) + E_x(u(X_{\tau_1})I_{X_{\tau_1}\notin B_1}). \tag{4.15}$$

We first prove that there exists constant $l_0>0$ such that

$$E_x(u(X_{\tau_1})I_{X_{\tau_1}\notin B_1}) \le u(x_0), \quad x\in G'_{l_0}\cap B(Q,1/4). \tag{4.16}$$

Denote the Green function of (X_t) on an open set U by G^U. For $x\in G'_{1/8}\cap B(Q,1/4)$ satisfying

$$|x-y|\le 2|z-y|, \qquad z\in B_0, \quad y\notin B_1,$$

we have by Proposition 3.2 and the last conclusion in Lemma 3.4

$$\begin{aligned}
&E_x(u(X_{\tau_1})I_{X_{\tau_1}\notin B_1})\\
=&\mathcal{A}(n,-\alpha)\int_{B_0}\int_{y\in G,|y-x|>\rho(x)^\gamma}\frac{\kappa(z,y)G^{B_0}(x,z)}{|z-y|^{n+\alpha}}u(y)dzdy\\
\le&2^{n+\alpha}\mathcal{A}(n,-\alpha)\int_{B_0}G^{B_0}(x,z)\int_{y\in G,|y-x|>\rho(x)^\gamma}\frac{C_2u(y)}{|x-y|^{n+\alpha}}dzdy\\
=&2^{n+\alpha}\mathcal{A}(n,-\alpha)(E_x\tau_1)\int_{y\in G,|y-x|>\rho(x)^\gamma}\frac{C_2u(y)}{|x-y|^{n+\alpha}}dy\\
\le&2^{n+2\alpha}C_2A'_6\mathcal{A}(n,-\alpha)\rho(x)^\alpha\Big(\int_{y\in G,|y-x|>\rho(x)^\gamma,|y-x_0|>2\rho(x_0)/3}\frac{u(y)}{|x-y|^{n+\alpha}}dy\\
+&\int_{|y-x_0|\le 2\rho(x_0)/3}\frac{u(y)}{|x-y|^{n+\alpha}}dy\Big)\\
:=&2^{n+2\alpha}C_2A'_6\mathcal{A}(n,-\alpha)\rho(x)^\alpha(I_1+I_2).
\end{aligned}\tag{4.17}$$

Similarly,

$$\begin{aligned}
u(x_0)\ge&E_{x_0}(u(X_{\tau_2})I_{X_{\tau_2}\notin B_3})\\
\ge&2^{-(n+\alpha)}C_1\mathcal{A}(n,-\alpha)\int_{B_2}G^{B_2}(x,z)\int_{y\in G,|y-x|>2\rho(x_0)/3}\frac{u(y)}{|x_0-y|^{n+\alpha}}dzdy\\
\ge&2^{-(n+\alpha)}C_1A_6\mathcal{A}(n,-\alpha)(\rho(x_0)/3)^\alpha\int_{y\in G,|y-x|>2\rho(x_0)/3}\frac{u(y)}{|x_0-y|^{n+\alpha}}dy.
\end{aligned}\tag{4.18}$$

We have $|y-x|\ge 2^{-1}\rho(x)^\gamma|y-x_0|$ if $|y-x|\ge\rho(x)^\gamma$ and $x\in B(Q,1/4)$. This and (4.18) show that

$$\begin{aligned}
I_1\le&2^{n+\alpha}\rho(x)^{-\gamma(n+\alpha)}\int_{y\in G,|y-x_0|\ge 2\rho(x_0)/3}\frac{u(y)}{|x_0-y|^{n+\alpha}}dy\\
\le&2^{2(n+\alpha)}3^\alpha A'_6(C_1\mathcal{A}(n,-\alpha)\rho(x_0)^\alpha)^{-1}\rho(x)^{-\gamma(n+\alpha)}u(x_0).
\end{aligned}\tag{4.19}$$

On the other hand, if $\rho(x)<\rho(x_0)/6$, we have by Harnack inequality (3.14)

$$\begin{aligned}
I_2\le&\int_{|y-x_0|\le 2\rho(x_0)/3}\frac{u(y)}{|x-y|^{n+\alpha}}dy\\
\le&2^{n+\alpha}A_7\int_{|y-x|>\rho(x_0)/6}\frac{u(x_0)}{|x-y|^{n+\alpha}}dy\\
\le&(2\pi)^n2^{n+\alpha}A_7(\rho(x_0)/6)^{-\alpha}u(x_0).
\end{aligned}\tag{4.20}$$

Combing (4.17)-(4.20), we have for some constant $c = c(n, \alpha, \beta, \Lambda, C_0, C_1, C_2)$

$$E_x(u(X_{\tau_1}) I_{X_{\tau_1} \notin B_1}) \leq c\rho(x)^{\alpha - \gamma(n+\alpha)} u(x_0), \qquad x \in G_{\rho(x_0)/6} \cap B(Q, 1/4). \tag{4.21}$$

Noticing that $\alpha - \gamma(n + \alpha) > 0$, by choosing $l_0 = l_0(n, \alpha, \beta, \Lambda, C_0, C_1, C_2)$ small enough, we get (4.16) from (4.21).

Suppose that there exists $x_1 \in G \cap B(Q, 1/8)$ such that $u(x_1) \geq M = M(n, \alpha, \beta, \Lambda, C_0, C_1, C_2) > l_0^{-\beta'} \vee (1+\delta^{-1})$ (M will be fixed later). By (4.14), $M > l_0^{-\beta'}$ and $u(x_0) = 1$ we have $\rho(x_1) < l_0$. By (4.15),(4.16) and $M > 1 + \delta^{-1}$,

$$E_{x_1}(u(X_{\tau_1}) I_{X_{\tau_1} \in B_1}) \geq \frac{1}{1+\delta} M.$$

From this inequality and (4.13) we can find $x_2 \in G$ such that

$$|x_1 - x_2| \leq \rho(x_1)^\gamma, \quad u(x_2) > (1 - \delta^2)^{-1} M.$$

Inductively, if $x_k \in G \cap B(Q, 1/4)$ for some $k \geq 2$, we can find $x_{k+1} \in G$ such that

$$|x_{k+1} - x_k| \leq \rho(x_k)^\gamma, \quad u(x_{k+1}) > (1 - \delta^2)^{-1} u(x_k) > (1 - \delta^2)^{-k} M. \tag{4.22}$$

By (4.14) and (4.22), we have $\rho(x_k) \leq (1 - \delta^2)^{k/\beta'} M^{-1/\beta'}$. Therefore, if (4.22) holds, we have

$$|x_k| \leq |x_1| + \sum_{i=1}^{k-1} |x_{i+1} - x_i| \leq 1/8 + (1 - (1 - \delta^2)^{1/\beta'})^{-1} M^{-1/\beta'}.$$

Thus for $M = (l_0^{-\beta'} \vee (1 + \delta^{-1})) \vee (8^{\beta'}(1 - (1 - \delta^2)^{1/\beta'})^{-\beta'})$, we can find $x_k \in G \cap B(Q, 1/4)$ satisfying (4.22) for all $k \geq 1$. This gives a contradiction by noticing that $\lim_{k \to \infty} u(x_k) = \infty$ and that u vanishes on $\partial G \cap B(Q, 1)$ continuously. Therefore $\sup_{y \in G \cap B(Q,1/8)} u(y) \leq M$. $\square$

Remark 4.1. *Let G and κ satisfy the conditions in Theorem 1.1, then we can prove the hitting probability estimates in Lemma 4.1 and the Carleson estimate in Lemma 4.2 still hold. This is due to that we have the same (super,sub) harmonic functions for $\kappa = \frac{|x-y|^{n+\alpha}}{|x-\overline{y}|^{n+\alpha}}$ and $\kappa \equiv 1$ (see (3.1)) and the term $C'|x - y|$ does not destroy the (super,sub) harmonic functions which we construct above (c.f. Lemma 2.3). We omit the proof of this extension because it can be done by following the arguments for $\kappa \in C^1(\overline{G} \times \overline{G})$. Notice that function κ in Theorem 1.1 satisfies condition $[C_1, C_2, C_3, -1]$ for some constant C_1, C_2, C_3, and hence the Harnack inequality in Theorem 3.5 holds.*

Before proving Theorem 1.1, we give some remarks on the assumptions of κ and G. When $\psi_2 \neq 0$, the condition of κ in (1.5) is to study the reflected subordinate Brownian motion. However, due to the definition of the reflection point, we need $C^{1,1}$ condition on G in Theorem 1.1 when $\psi_2 \neq 0$. Let G be a C^2 open set in $\mathbb{R}^n$. By the Appendix in [22], there exists $\delta_0 > 0$ such that, for any $x \in G'_{\delta_0}$, there is an unique point $\xi(x) \in \partial G$ satisfying $|x - \xi(x)| = \rho(x)$, $\xi \in C^1(\overline{G'_{\delta_0}})$ and $\rho \in C^2(\overline{G'_{\delta_0}})$. For $x \in G'_{\delta_0}$, define the reflection point of x by

$$\overline{x} = 2\xi(x) - x. \tag{4.23}$$

When G is a $C^{1,1}$ open set, ξ and ρ are Lipschitz and $C^{1,1}$ in a neighborhood of ∂G, respectively. The proof for the uniqueness of $\xi(x)$ is similar to [22]. The Lipschitz and $C^{1,1}$ properties follow by the C^2 case and the standard smooth approximation.

Proof of Theorem 1.1: First we assume that G is a $C^{1,\beta-1}$ open set with characteristics $r_0 < 1, \Lambda$ and κ satisfies the conditions in Proposition 2.4. Let $u \geq 0$ be a function on G which is not identical to zero, harmonic on $G \cap B(Q,r)$ and vanishes continuously on $\partial G \cap B(Q,r)$ for some $0 < r < r_0$. By scaling and translation we can assume that $r = 1$ and $Q = 0$. Take the coordinate system CS_0 and denote

$$B_0 = \triangle(0, A_8, A_8), \qquad B_1 = \triangle(0, 2A_8, 2A_8), \qquad \tau = \tau_{B_0}.$$

By scaling, we can also assume that

$$B_1 \subseteq B(0, 1/3). \tag{4.24}$$

Write

$$x_0 = (0, 1/2), \quad x_1 = (0, 3A_8/2).$$

By Harnack inequality (3.14), we have

$$k_1^{-1} u(x_0) \leq u(x) \leq k_1 u(x_0), \quad x \in \triangle(0, 2A_8, A_8) \setminus \triangle(0, A_8, A_8) \tag{4.25}$$

for some constant $k_1 = k_1(n, \alpha, \Lambda, C_1, C_2, M)$. Next we assume that $x \in \triangle(0, A_8, A_8)$ with $\widetilde{x} = 0$. Since u is harmonic on $G \cap B(Q,1)$, we have by (4.2) and (4.25)

$$\begin{aligned} u(x) =& E_x u(X_\tau) \geq k_1^{-1} u(x_0) P_x(u(X_\tau) \in \triangle(0, 2A_8, A_8)) \\ \geq& A_9^{-1} k_1^{-1} u(x_0) \rho(x)^{\alpha-1}. \end{aligned} \tag{4.26}$$

By the same calculation as (4.18), we have

$$u(x_1) \geq k_2 \int_{y \notin B_1} \frac{u(y)}{|x_1 - y|^{n+\alpha}} dy \tag{4.27}$$

for some $k_2 = k_2(n, \alpha, C_1, C_2, M)$. By (4.24) and Proposition 3.2, we can also find a constant $k_3 = k_3(n, \alpha, \Lambda)$ such that (c.f. (3.18))

$$E_x(\tau) \leq k_3 P_x(X_\tau \in G \setminus B_1). \tag{4.28}$$

By definition of x_1, we can find a constant $k_4 = k_4(\Lambda)$ such that $|z - y| \geq k_4 |x_1 - y|$ for $z \in B_0$ and $y \notin B_1$. Thus, by Proposition 3.2, Lemmas 3.4, 4.1, 4.2 and applying (4.24), (4.25), (4.27) and (4.28), we have

$$\begin{aligned} u(x) =& \mathcal{A}(n, -\alpha) \int_{B_0} \int_{y \in G \cap B_1} \frac{\kappa(z,y) G^{B_0}(x,z)}{|z - y|^{n+\alpha}} u(y) dz dy \\ &+ \mathcal{A}(n, -\alpha) \int_{B_0} \int_{y \notin G \cap B_1} \frac{\kappa(z,y) G^{B_0}(x,z)}{|z - y|^{n+\alpha}} u(y) dz dy \\ &\leq C_2 \mathcal{A}(n, -\alpha) A_{10} u(x_0) P_x(X_\tau \in B_1) + C_2 \mathcal{A}(n, -\alpha) k_4^{-(n+\alpha)} \int_{B_0} \int_{y \notin B_1} \frac{G^{B_0}(x,z)}{|x_1 - y|^{n+\alpha}} u(y) dz dy \\ &\leq C_2 \mathcal{A}(n, -\alpha) A_9 A_{10} u(x_0) \rho(x)^{\alpha-1} + C_2 \mathcal{A}(n, -\alpha) k_4^{-(n+\alpha)} E_x(\tau) \int_{y \notin B_1} \frac{u(y)}{|x_1 - y|^{n+\alpha}} dy \\ &\leq C_2 \mathcal{A}(n, -\alpha) u(x_0) A_9 (A_{10} + k_1 k_2^{-1} k_3 k_4^{-(n+\alpha)}) \rho(x)^{\alpha-1}. \end{aligned} \tag{4.29}$$

Combing (4.26) and (4.29) we prove (1.6) for $x \in \triangle(0, A_8, A_8)$ with $\widetilde{x} = 0$. By considering the coordinate system CS_y for $y \in B(Q, 2/3) \cap \partial G$, applying the arguments above and the Harnack inequality (3.14), we can prove (1.6). The general case can be proved similarly with the help of Remark 4.1.

□

5 Boundary Harnack inequality of $\Delta_G^{\alpha/2,\kappa}$ on Lipschitz domain

To simplify notations, we assume that $\kappa \equiv 1$ in the arguments below because the estimates are the same for the general cases. Let G be a Lipschitz domain with characteristic r_0 and Λ, i.e., for each $x_0 \in \partial G$, we can find a Lipschitz function $\Gamma_{x_0} : \mathbb{R}^{n-1} \to \mathbb{R}$ with Lipschitz coefficient not greater than Λ and an orthonormal coordinate system CS_{x_0} with which it holds that

$$G \cap B\big(x_0, r_0\big) = \{y = (y_1, \cdots, y_n) : \ y_n > \Gamma_{x_0}(y_1, \cdots y_{n-1}) \ \} \cap B\big(x_0, r_0\big). \tag{5.1}$$

The following hitting probability estimate is obvious for the Brownian motion case. Here we use capacity to give this estimate. We refer to [21] for more details about capacity and energy measure class S_{00} of symmetric Markov processes.

Lemma 5.1. *Let G be a Lipschitz domain with characteristic $r_0 = 1$ and Λ. Let $x_0 = 0 \in \partial G$ and choose the coordinate system CS_0. Assume that A is a constant such that $\triangle(0, A, A) \subset G \cap B(0, 1)$. Then there exists a constant $A_{11} = A_{11}(n, \alpha, \Lambda)$ such that*

$$P_x(X_\tau \in \partial G) \geq 1/A_{11}, \qquad x = (\widetilde{0}, x_n), \quad 0 < x_n < A/2, \tag{5.2}$$

where $\tau = \inf\{t > 0 : X_t \in \triangle(0, A, A)^c\}$.

Proof By scaling, we may assume that $A > 1/(3(1 + \Lambda))$ without loss of generality. Denote the heat kernel of (X_t) by $p(t, x, y)$. By Theorem 1.1 in [17], there exists constant $k_1 = k_1(n, \alpha, \Lambda)$ such that

$$k_1\big(t^{-n/\alpha} \wedge \frac{t}{|x - y|^{n+\alpha}}\big) \leq p(t, x, y) \leq k_1^{-1}\big(t^{-n/\alpha} \wedge \frac{t}{|x - y|^{n+\alpha}}\big), \quad 0 < t < 1. \tag{5.3}$$

Set $F = \overline{\triangle(0, A, A)^c \cap G}$ and denote by (Y_t) the killed process of (X_t) when hitting F. We know that the heat kernel $p^0(t, x, y)$ of (Y_t) is given by

$$p^0(t, x, y) = p(t, x, y) - \int_0^t \int_F p(t - s, z, y) P_x(X_\sigma \in dz, \sigma \in ds), \tag{5.4}$$

where $\sigma = \inf\{t > 0 : X_t \in F\}$. Noticing that $A > 1/(3(1 + \Lambda))$ and choosing δ small enough we get by (5.3) and (5.4)

$$p^0(t, x, y) \geq k_2\big(t^{-n/\alpha} \wedge \frac{t}{|x - y|^{n+\alpha}}\big), \quad 0 < t < 1, \ x, y \in \overline{G} \cap B(0, \delta) \tag{5.5}$$

for some $k_2 = k_2(n, \alpha, \Lambda)$. Set $\Gamma = \partial G \cap \overline{B(0, \delta)}$. Define the 1-potential kernel of (Y_t) by $U_1^0(x, y) = \int_0^\infty e^{-t} p^0(t, x, y)\, dt$ and define for measure μ on $\overline{G} \setminus F$

$$U_1^0 \mu(x) = \int_{\overline{G} \setminus F} U_1^0(x, y) \mu(dy). \tag{5.6}$$

By Theorem 4.2.5 in [21] and the continuity argument, there exists a 1-equilibrium measure ν_Γ supported on Γ such that

$$U_1^0 \nu_\Gamma(x) = E_x^0(e^{-\sigma_\Gamma}), \qquad x \in \overline{G} \setminus F, \tag{5.7}$$

where $\sigma_\Gamma = \inf\{t > 0 : Y_t \in \Gamma\}$. By problem 2.2.2 in [21],

$$\nu_\Gamma(\Gamma) = \sup\{\mu(\Gamma) : \mu \in S_{00}, \ supp[\mu] \subseteq K, \ U_1^0 \mu \leq 1\}. \tag{5.8}$$

Direct calculations shows that $\mu = \delta I_\Gamma m(dx) \in S_{00}$ for $\delta > 0$. Choosing δ small enough and applying the second inequality in (5.3), we get $\nu_\Gamma(\Gamma) > k_3(n, \alpha, \Lambda)$. Therefore by (5.5) and (5.7), we have for $x \in \overline{G} \cap B(0, \delta)$

$$
\begin{aligned}
E_x^0(e^{-\sigma_\Gamma}) \geq & e^{-1} \int_\Gamma \int_0^1 p_0(t, x, y) dt \nu_\Gamma(dy) \\
\geq & e^{-1} \nu_\Gamma(\Gamma) \inf_{y \in \Gamma} \int_0^1 p_0(t, x, y) dt \geq k_4(n, \alpha, \Lambda),
\end{aligned} \tag{5.9}
$$

which implies that

$$
P_x^0(\sigma_\Gamma < \infty) \geq E_x^0(e^{-\sigma_\Gamma}) \geq k_4. \tag{5.10}
$$

Noticing that $P_x(X_\tau \in \partial G) \geq P_x^0(\sigma_\Gamma < \infty)$, we get (5.2) for $x \in \overline{G} \cap B(0, \delta)$. Thus we complete the proof by the Harnack inequality in Theorem 3.5. $\square$

Let A be a constant such that $\triangle(0, 2A, 2A) \subset G \cap B(0, 1)$ under the coordinate system CS_0 for a Lipschitz domain G with characteristic $r_0 = 1$ and Λ. Set

$$
K_0 = \triangle(0, A, A), \quad K_1 = \triangle(0, 2A, A) \setminus K_0, \quad K_2 = G \setminus \triangle(0, 2A, A); \tag{5.11}
$$

$$
H_1 = \{X_{\tau_{K_0}} \in K_1\}, \qquad H_2 = \{X_{\tau_{K_0}} \in K_2\}. \tag{5.12}
$$

Lemma 5.2. *With notations defined in (5.11) and (5.12), for any $k \geq 0$, there exists a constant $A_{12} = A_{12}(n, \alpha, \Lambda, A, k)$ such that*

$$
P_y(H_1) \geq A_{12} \rho(y)^\alpha |\ln \rho(y)|^k, \qquad y \in K_0, \ |\widetilde{y}| \leq A/2. \tag{5.13}
$$

Proof Let $y \in K_0$ with $|\widetilde{y}| \leq A/2$ and $\tau = \tau_{B(y, \rho(y)/2)}$. We assume also that $B(y, \rho(y)/2) \subset K_0$. Otherwise (5.13) can be verified by showing that $P_y(H_1) > k_1(n, \alpha, \Lambda, A)$. As the calculations in (3.18), we have by Lemma 3.4

$$
\begin{aligned}
P_y(H_1) \geq & k_2(n, \alpha) E_y(\tau) \int_{z \in K_1} \frac{1}{|z - y|^{n+\alpha}} dz \\
\geq & k k_3(n, \alpha, \Lambda, A) \rho(y)^\alpha,
\end{aligned} \tag{5.14}
$$

which gives (5.13) for $k = 0$. Suppose (5.13) holds for some $k \geq 0$. By the strong Markov property

$$
\begin{aligned}
P_y(H_1) \geq & P_y(\tau < \tau_{K_0}, \ X_{\tau_{K_0}} \in K_1) \\
= & \int_{K_0} P_z(H_1) P_y(X_\tau \in dz) \\
\geq & k_4(n, \alpha) \int_{z \in K_0 \setminus B(y, \rho(y)/2)} P_z(H_1) \frac{E_y(\tau)}{|z - y|^{n+\alpha}} dz \\
\geq & k_5(n, \alpha, \Lambda, A, k) \int_{z \in K_0 \setminus B(y, \rho(y)/2), |\widetilde{z}| \leq A/2} \frac{\rho(y)^\alpha \rho(z)^\alpha |\ln \rho(z)|^k}{|z - y|^{n+\alpha}} dz.
\end{aligned} \tag{5.15}
$$

Direct calculation shows that $P_y(H_1) \geq A'_{12} \rho(y)^\alpha |\ln \rho(y)|^{k+1}$. Hence the proof completes by induction. $\square$

Lemma 5.3. *With notations defined in (5.11) and (5.12), there exists a constant $A_{13} = A_{13}(n, \alpha, \Lambda)$ such that*

$$
P_y(H_2) \leq A_{13} P_y(H_1), \qquad y \in K_0, \ \widetilde{y} = 0. \tag{5.16}
$$

Proof To simplify the arguments, we assume that G is a special Lipschitz domain. By scaling, we assume that $A=1$ in (5.11). For $i\geq 1$, set

$$J_i=\triangle(0,2^{-i},r_i)\setminus\triangle(0,2^{-i-1},r_i),\qquad r_i=\frac{1}{2}-\frac{1}{50}\sum_{j=1}^{i}\frac{1}{j^2},$$

and $r_0=r_1$. Define for $i\geq 1$

$$d_i=\sup_{z\in J_i}P_z(H_2)/P_z(H_1),\quad \widetilde{J}_i=\triangle(0,2^{-i},r_{i-1}),\qquad \tau_i=\tau_{\widetilde{J}_i}. \tag{5.17}$$

By Harnack inequality, each d_i is finite. Noticing that $\tau_i\leq\tau_{K_0}$ and applying the strong Markov property, we have for $z\in J_i$ and $i\geq 2$

$$\begin{aligned}P_z(H_2)=&P_z(X_{\tau_{K_0}}\in K_2,\ X_{\tau_i}\in\cup_{k=1}^{i-1}J_k)+P_z(X_{\tau_{K_0}}\in K_2,\ X_{\tau_i}\in G\setminus\cup_{k=1}^{i-1}J_k)\\ \leq&\sum_{k=1}^{i-1}\int_{J_k}P_z(X_{\tau_i}\in dw)P_w(H_2)+P_z(\ X_{\tau_i}\in G\setminus\cup_{k=1}^{i-1}J_k)\\ \leq&\sum_{k=1}^{i-1}d_k\int_{J_k}P_z(X_{\tau_i}\in dw)P_w(H_1)+P_z(\ X_{\tau_i}\in G\setminus\cup_{k=1}^{i-1}J_k)\\ \leq&(\sup_{1\leq k\leq i-1}d_k)P_z(H_1)+P_z(X_{\tau_i}\in G\setminus\cup_{k=1}^{i-1}J_k).\end{aligned} \tag{5.18}$$

Define $\sigma_0=0,\sigma_1=\inf\{t>0:|X_t-X_0|\geq 2^{-i}\}$ and define by induction $\sigma_{m+1}=\sigma_1\circ\theta_{\sigma_m}$ for $m\geq 1$. Similar to the calculation of (3.18), we have for some constant $k_1<1$ independent of i such that

$$P_w(X_{\sigma_1}\in\widetilde{J}_i)\leq 1-P_w(X_{\sigma_1}\in\cup_{k=1}^{i-1}J_k)<k_1,\quad w\in\widetilde{J}_i.$$

Therefore, for $z\in J_i$ and positive integer l, we have by the strong Markov property for

$$\begin{aligned}P_z(\tau_i>\sigma_{li})\leq&P_z(X_{\sigma_k}\in\widetilde{J}_i,1\leq k\leq li)\\ =&\int_{w\in\widetilde{J}_i}P_z(X_{\sigma_k}\in\widetilde{J}_i,1\leq k\leq li-2,X_{\sigma_{li-1}}\in dw)P_w(X_{\sigma_1}\in\widetilde{J}_i)\\ \leq&P_z(X_{\sigma_k}\in\widetilde{J}_i,1\leq k\leq li-1)k_1\leq k_1^{li}.\end{aligned} \tag{5.19}$$

Recall that $\widetilde{x}$ is the first $n-1$ coordinate of x. On $\{X_{\tau_i}\in G\setminus\cup_{k=1}^{i-1}J_k,\ \ \tau_i\leq\sigma_{li}\}$ with $X_0=z\in J_i$, we have $|\widetilde{X}_{\sigma_k}-\widetilde{X}_{\sigma_0}|>\frac{1}{50i^2}-2^{-i}$ for some $1\leq k\leq li$ which implies for some $1\leq k'\leq li$

$$|X_{\sigma_{k'}}-X_{\sigma_{k'-1}}|\geq(\frac{1}{50i^2}-2^{-i})/(li).$$

Therefore, we have for some $i_0\geq 2$

$$\begin{aligned}&\{X_{\tau_i}\in G\setminus\cup_{k=1}^{i-1}J_k,\quad\tau_i\leq\sigma_{li}\}\\ \subseteq&\cup_{k=1}^{li}\{|X_{\sigma_k}-X_{\sigma_{k-1}}|\geq 1/(100li^3),X_{\sigma_{k-1}}\in\widetilde{J}_i\},\quad i\geq i_0,\end{aligned} \tag{5.20}$$

and hence

$$\begin{aligned}&P_z(X_{\tau_i}\in G\setminus\cup_{k=1}^{i-1}J_k,\quad\tau_i\leq\sigma_{li})\\ \leq&\sum_{k=1}^{li}P_z(|X_{\sigma_k}-X_{\sigma_{k-1}}|\geq 1/(100li^3),X_{\sigma_{k-1}}\in\widetilde{J}_i)\end{aligned}$$

$$
\begin{aligned}
&\le li \sup_{z\in\widetilde{J}_i} P_z(|X_{\sigma_1}| \ge 1/(100li^3)) \\
&\le k_2 li 2^{-\alpha i}(li^3)^\alpha. \qquad (5.21)
\end{aligned}
$$

The proof of the last inequality above is similar to (4.17). By (5.19), (5.21) and choosing l big enough, we have for $z \in J_i$ and $i \ge i_0$,

$$
P_z(X_{\tau_i} \in G \setminus \cup_{k=1}^{i-1} J_k) \le k_1^{li} + k_2 li 2^{-\alpha i}(li^3)^\alpha \le k_3 2^{-\alpha i} i^{3\alpha+1}. \qquad (5.22)
$$

By (5.18), (5.22) and Lemma 5.2, for $z \in J_i$ and $i \ge i_0$

$$
P_z(H_2)/P_z(H_1) \le \sup_{1\le k\le i-1} d_k + P_z(X_{\tau_i} \in G \setminus \cup_{k=1}^{i-1} J_k)/P_z(H_1) \le \sup_{1\le k\le i-1} d_k + k_4/i^2.
$$

This implies that

$$
d_i \le \sup_{1\le k\le i_0-1} d_k + k_4 \sum_{k=1}^{i} 1/k^2 \le \sup_{1\le k\le i_0-1} d_k + 3k_4,
$$

which completes the proof of this lemma. □

Remark 5.1. *One may use the method in [10] to give a better estimate of (5.13). The proof of Lemma 5.3 is an adaption of the Brownian motion case.*

Proof of Theorem 1.2: In the proof of Lemma 4.2, we only use the $C^{1,\beta-1}$ property in (4.13). Thus we can prove the Carleson estimate for the Lipschitz case with Lemma 5.1 in place of (4.13). Therefore, we can prove Theorem 1.2 by the standard arguments of BHI with the help of Theorem 3.5 and Lemma 5.3. □

6 Boundary Harnack inequality of $\Delta^{\alpha/2}$

When $G = \mathbb{R}^n$, $\Delta_G^{\alpha/2}$ is the fractional Laplacian $\Delta^{\alpha/2}$. Recall that $w_p(y) = y_n^p$ for $y \in \mathbb{R}^n_+$. We extend these functions to $\mathbb{R}^n$ by taking zero on $\mathbb{R}^n \setminus \mathbb{R}^n_+$. Next we give a formula of $\Delta^{\alpha/2} w_p$. Integration by parts formula shows that for $0 < p < \alpha$

$$
\begin{aligned}
&\int_0^\infty \frac{y^p - x^p}{|y-x|^{1+\alpha}} dy = x^{p-\alpha} \int_0^\infty \frac{y^p - 1}{|y-1|^{1+\alpha}} dy \\
=& \lim_{\varepsilon\downarrow 0} x^{p-\alpha} \left(\int_0^{1-\varepsilon} \frac{y^p - 1}{|y-1|^{1+\alpha}} dy + \int_{1+\varepsilon}^\infty \frac{y^p - 1}{|y-1|^{1+\alpha}} dy \right) \\
=& \frac{1}{\alpha} x^{p-\alpha} + \frac{p}{\alpha} x^{p-\alpha} \int_0^1 \frac{y^{\alpha-p-1} - y^{p-1}}{|y-1|^\alpha} dy. \qquad (6.1)
\end{aligned}
$$

Thus, for $n = 1$ we have

$$
\Delta^{\alpha/2} w_p(x) = \mathcal{A}(1, -\alpha) \frac{p}{\alpha} x^{p-\alpha} \int_0^1 \frac{y^{\alpha-p-1} - y^{p-1}}{|y-1|^\alpha} dy, \quad 0 < p < \alpha. \qquad (6.2)
$$

Applying spherical coordinate transform from $(y_1, \cdots, y_n) \in \mathbb{R}^n$ to $(r, \theta_1, \cdots, \theta_{n-1}) \in [0,\infty) \times [0,\pi]^{n-2} \times [0, 2\pi)$, this gives for $n > 1$

$$
\begin{aligned}
&\mathcal{A}(n, -\alpha)^{-1} \Delta^{\alpha/2} w_p(x) \\
=& \lim_{\varepsilon\downarrow 0} \int_{|y-x|>\varepsilon} \frac{w_p(y) - w_p(x)}{|x-y|^{n+\alpha}} dy
\end{aligned}
$$

$$
\begin{aligned}
=&\lim_{\varepsilon\downarrow 0}\int_0^{\pi/2} d\theta_1\cdots\int_0^{\pi} d\theta_{n-2}\int_0^{2\pi}\varphi(\theta_1,\cdots\theta_{n-2})\ d\theta_{n-1}\cdot\\
&\int_{-\infty}^{\infty} I_{\{|t-\frac{x_n}{\cos\theta}|>\varepsilon\}}\cos^p\theta_1\frac{t^pI_{t\geq 0}-(\frac{x_n}{\cos\theta_1})^p}{|t-\frac{x_n}{\cos\theta_1}|^{1+\alpha}}\ dt\\
=&\frac{p}{\alpha}\int_0^1\frac{y^{\alpha-p-1}-y^{p-1}}{|y-1|^{\alpha}}dy\int_0^{\pi/2} d\theta_1\cdots\int_0^{\pi} d\theta_{n-2}\int_0^{2\pi}\varphi(\theta_1,\cdots\theta_{n-2})\ (\cos^{\alpha}\theta_1)x_n^{p-\alpha}d\theta_{n-1}\\
=&\frac{p}{\alpha}\int_0^1\frac{y^{\alpha-p-1}-y^{p-1}}{|y-1|^{\alpha}}dy\int_{|y|=1,y_n\geq 0} y_n^{\alpha}\ m(dy)\cdot w_{p-\alpha}(x),
\end{aligned}\tag{6.3}
$$

where $\varphi(\theta_1,\cdots\theta_{n-2})=\sin^{n-2}\theta_1\sin^{n-3}\theta_2\cdots\sin\theta_{n-2}$, $m(dy)$ is the $(n-1)$-dimensional Lebesgue measure and we use the following transform in the calculation above

$$
r=t-x_n/\cos\theta_1,\quad \theta_1\in[0,\pi/2);\qquad -r=t+x_n/\cos\theta_1,\quad \theta_1\in(\pi/2,\pi].
$$

Denote for $0<p<\alpha$ and $n\geq 1$

$$
\begin{aligned}
\Lambda(n,\alpha,p)=&\frac{p\mathcal{A}(n,-\alpha)}{\alpha}\int_0^1\frac{y^{\alpha-p-1}-y^{p-1}}{|y-1|^{\alpha}}dy\int_{|y|=1,y_n\geq 0} y_n^{\alpha}\ m(dy),\\
\overline{\Lambda}(n,\alpha,p)=&\frac{\mathcal{A}(n,-\alpha)}{\alpha}\left(1+p\int_0^1\frac{y^{\alpha-p-1}-y^{p-1}}{|y-1|^{\alpha}}dy\right)\int_{|y|=1,y_n\geq 0} y_n^{\alpha}\ m(dy)
\end{aligned}
$$

with convention that $m(dy)$ is the Dirac measure for $n=1$. By (6.3), we have the following Lemma. We notice that the case $p=\alpha/2$ below has been obtained in Example 3.2 of [7].

Lemma 6.1. *Let $0<p<\alpha<2$, we have*

$$
\Delta^{\alpha/2}w_p=\Lambda(n,\alpha,p)w_{p-\alpha},\qquad x\in\mathbb{R}^n_+,\tag{6.4}
$$

$$
\Delta^{\alpha/2}_{\mathbb{R}^n_+}w_p=\overline{\Lambda}(n,\alpha,p)w_{p-\alpha},\qquad x\in\mathbb{R}^n_+.\tag{6.5}
$$

Formula (6.5) is another version of (3.1) for $0<p<\alpha$. By Lemma 6.1 and (3.1) we see that

$$
\Delta^{\alpha/2}w_p<0,\quad -1<p<\alpha/2;\quad \Delta^{\alpha/2}w_p=0,\quad p=\alpha/2;\quad \Delta^{\alpha/2}w_p>0,\quad \alpha/2<p<\alpha.\tag{6.6}
$$

Let κ be a symmetric function on $\mathbb{R}^n\times\mathbb{R}^n$ such that

$$
R_1<\kappa(x,y)<R_2,\quad |\kappa(x,y)-\kappa(x,x)|\leq R_3|x-y|,\quad x,y\in\mathbb{R}^n\tag{6.7}
$$

for some constants $R_1,R_2,R_3>0$. In what follows, notation (X_t) is for the stable-like process on $\mathbb{R}^n$ associated with $\Delta^{\frac{\alpha}{2},\kappa}$. Harmonic functions of (X_t) is again defined by (2.1).

Lemma 6.2. *Let $0<\alpha\leq 1\vee\alpha<\beta\leq 2$ and D a $C^{1,\beta-1}$ open set in $\mathbb{R}^n$ with characteristics $r_0=1$ and Λ. Let κ be a symmetric function on $\mathbb{R}^n\times\mathbb{R}^n$ satisfying (6.7). Then for $\alpha/2\leq p<\alpha$ and $Q\in\partial D$, there exist function u_p and constants $A_{13}=A_{13}(\Lambda)$, $A_{14}=A_{14}(n,\alpha,\beta,\Lambda,p,R_1,R_2,R_3)$ such that*

$$
A_{13}^{-1}I_{D\cap B(Q,2/3)}\rho(x)^p\leq u_p(x)\leq A_{13}I_{D\cap B(Q,2/3)}\rho(x)^p,\quad x\in\mathbb{R}^n,\tag{6.8}
$$

$$
\Delta^{\frac{\alpha}{2},\kappa}u_p(x)\geq A_{14}\rho(x)^{p-\alpha},\qquad x\in D\cap B(Q,1/A_{14}),\quad \alpha/2<p<\alpha,\tag{6.9}
$$

and

$$
|\Delta^{\frac{\alpha}{2},\kappa}u_{\alpha/2}(x)|\leq\begin{cases} A_{14}\rho(x)^{\beta-\alpha/2-1}, & x\in D\cap B(Q,1/2),\quad \beta<1+\alpha/2,\\ A_{14}|\log\rho(x)|, & x\in D\cap B(Q,1/2),\quad \beta=1+\alpha/2,\\ A_{14}, & x\in D\cap B(Q,1/2),\quad \beta>1+\alpha/2.\end{cases}\tag{6.10}
$$

Proof Following the calculations in Lemma 2.1, Lemma 2.2 and Proposition 2.4 we can prove this lemma with the help of (6.4) and (6.6). We omit the details of the proof because, by noticing that $w_p = 0$ on $\mathbb{R}^n \setminus \mathbb{R}^n_+$, the calculation is essentially on D which is the same as the regional fractional Laplacian case. $\square$

By Lemma 6.2 and following the arguments in Proposition 4.1 and Theorem 1.1, we can prove the following results.

Lemma 6.3. *Let $0 < \alpha \leq 1 \vee \alpha < \beta \leq 2$ and D a $C^{1,\beta-1}$ open set in $\mathbb{R}^n$ with characteristics $r_0 = 1$ and Λ. Assume that $Q = 0 \in \partial D$ and κ is a symmetric function on $\mathbb{R}^n \times \mathbb{R}^n$ satisfying (6.7). Then there exist constants $A_{15} = A_{15}(n, \alpha, \beta, \Lambda, R_1, R_2, R_3) < 1/2$ and $A_{16} = A_{16}(n, \alpha, \beta, \Lambda, R_1, R_2, R_3)$ such that*

$$\begin{aligned} A_{16}^{-1}\rho(x)^{\alpha/2} \leq & P_x\{X_{\tau_{\triangle(0,A_{15},A_{15})}} \in \triangle(0, 2A_{15}, A_{15})\} \\ \leq & P_x\{X_{\tau_{\triangle(0,A_{15},A_{15})}} \in D\} \leq A_{16}\rho(x)^{\alpha/2} \end{aligned} \tag{6.11}$$

for $x \in \triangle(Q, A_{15}, A_{15})$ with $\widetilde{x} = 0$ under CS_Q.

Theorem 6.4. *Assume that α, β, D and κ satisfy the same conditions as in Lemma 6.3. Let $Q \in \partial D$ and $r \in (0, r_0)$. Assume that $u \geq 0$ is a function on D which is not identical to zero, harmonic on $D \cap B(Q, r)$ and vanishes on $D^c \cap B(Q, r)$. Then there exists constant $C = C(n, \alpha, \Lambda, R_1, R_2, R_3)$ such that*

$$\frac{u(x)}{u(y)} \leq C\frac{\rho(x)^{\alpha/2}}{\rho(y)^{\alpha/2}}, \quad for\ x, y \in D \cap B(Q, r/2). \tag{6.12}$$

Remark 6.1. *By taking $G = D$, all the conclusions in Section 3 can be extended to $\Delta^{\frac{\alpha}{2},\kappa}$ in a similar way, where the reflected stable process is replaced by the stable-like process. The Carleson estimate for $\Delta^{\frac{\alpha}{2},\kappa}$ can be proved by the same method as in Lemma 4.2. We remark that to prove the boundary Harnack principle of $\Delta^{\frac{\alpha}{2},\kappa}$ on open sets, we need the method in [8] to get the Carleson estimate, where the explicit Poisson kernel can be replaced by the sharp estimates as in [18]. Theorem 6.4 can be generalized to operator $\Delta_G^{\frac{\alpha}{2},\kappa}$ when we further assume that $D \subset \overline{D} \subset G$. The proof of this generalization is the same as the case $G = \mathbb{R}^n$ except that the constant depends also on the distance between D and ∂G.*

Remark 6.2. *Since $((w_p)_{p<1}, (w_p)_{p>1})$ w_1 is the (super, sub)harmonic function of Laplacian on half spaces, by the Harnack inequality in [30] and the method in this paper, we can prove the explicit BHI for $\Delta + \Delta^{\alpha/2}$ on $C^{1,1}$ open sets which gives $\rho(x)$ order decay for harmonic functions near the boundary. With the help of this fact we can prove that the Green functions of $\Delta + \Delta^{\alpha/2}$ and Δ are comparable on a $C^{1,1}$ bounded open set.*

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References

[1] A. Ancona, Principe de Harnack à la frontière et théorème de Fatou pour un opérateur elliptique dans un domaine lipschitzien. Ann. Inst. Fourier (Grenoble) **28**, No. 4 (1978), 169–213.

[2] R.F. Bass, Probabilistic Techniques in Analysis. Springer, New York, 1995.

[3] R. Banuelos, R. F. Bass and K. Burdzy, Hölder domains and the boundary Harnack principle, Duke Math. J. **64** (1991), 195-200.

[4] R.F. Bass and K. Burdzy, Probabilistic proof of the boundary Harnack principle. In seminar on stochastic processes, 1989, Progr. Probab. **18** (1990), 1–16, Birkhäuser.

[5] R.F. Bass and K. Burdzy, A boundary Harnack principle in twisted Hölder domains. Ann. of Math. **134**, No. 2 (1991), 253–276.

[6] R.F. Bass and D.A. Levin, Harnack inequalities for jump processes. Potential Anal. **17** (2002), no. 4, 375–388.

[7] R. Banuelos and K. Bogdan, Symmetric stable processes in cones. Potential Anal. **21** (2004), no. 3, 263–288.

[8] K. Bogdan, The boundary Harnack principle for the fractional Laplacian. Studia Math. (**123**)(1)(1997),43-80.

[9] K. Bogdan, Sharp estimates for the Green function in Lipschitz domains. J.Math.Anal.Appl. **243** (2000), 326-337.

[10] K. Bogdan and T. Byczkowski, Probabilistic proof of boundary Harnack principle for α-harmonic functions. Potential Anal. **11**, 135–156 (1999).

[11] K. Bogdan, T. Kulczycki and M. Kwasnicki, Estimates and structure of α-harmonic functions, Probab. Theory Related Fields **140** (2008), no. 3-4, 345–381.

[12] K. Bogdan, K. Burdzy and Z.Q. Chen, Censored stable processes. Probab. Theory Relat. Fields. **127** (2003), 89-152.

[13] L. Caffarelli, E. Fabes, S. Mortola, and S. Salsa, Boundary behavior of nonnegative solutions of elliptic operators in divergence form. Indiana Univ. Math. J. **30**, No. 4 (1981), 621–640.

[14] Z.Q. Chen and P. Kim, Green function estimates for censored stable processes. Probab. Theory Relat. Fields. **124** (2002), 295-610.

[15] Z.Q. Chen, P. Kim, R.M. Song, Heat kernel estimates for Dirichlet fractional Laplacian, Preprint, 2008.

[16] Z.Q. Chen, P. Kim, R.M. Song, Two-sided heat kernel estimates for censored stable-like processes, Preprint, 2008.

[17] Z.Q. Chen and T. Kumagai, Heat Kernel estimates for stable-like process on d-Sets. Stochastic Process Appl. **108** (2003), 27-62.

[18] Z.Q. Chen and R. Song, Estimates on Green functions and Poission kernels of symmetric stable processes. Math. Ann. **312** (1998), 465-601.

[19] B. Dahlberg, Estimates of harmonic measure. Arch. Rat. Mech. Anal. **65**, (1977) 275–288.

[20] E. Fabes, N. Garofalo, S. Marín-Malave and S. Salsa, Fatou theorems for some nonlinear elliptic equations. Rev. Mat. Iberoamericana **4** (1988), no. 2, 227–251.

[21] M. Fukushima, Y. Oshima and M. Takeda, Dirichlet form and symmetric Markov processes. Walter de Gruyter, Berlin 1994.

[22] D. Gilbarg and N. Trudinger, Elliptic partial differential equations of second order. 2nd edition, Springer-verlag, 1983.

[23] Q.Y. Guan, The integration by parts formula for regional fractional Laplacian. Comm. Math. Phy. **266**, (2006), no. 2, 289-329.

[24] Q.Y. Guan and Z.M. Ma, The reflected α-symmetric stable processes and regional fractional Laplacian. Probab. Theory Related Fields **134** (2006), no. 4, 649–694.

[25] Q.Y. Guan and Z.M. Ma, Boundary value problems for fractional Laplacian. Stochastics and Dynamics, **5** (2005), no. 3, 385–424.

[26] N. Jacob and R. Schilling, Some Dirichlet spaces obtained by subordinate reflected diffusions. Rev. Mat. Iberoamericanna, **15** (1999), 59-91.

[27] Y.Y. Li and L. Nirenberg, The Dirichlet problem for singularly perturbed elliptic equations. Comm. Pure Appl. Math. **51** (1998), no. 11-12, 1445–1490.

[28] R. Song and J.M. Wu, Boundary Harnack principle for symmetric stable processes. J. Funct. Anal. **168**(2) (1999), 403-427.

[29] R. Song and Z. Vondracek, Harnack inequality for some classes of Markov processes. Math. Z. **246**, No. 1-2 (2004), 177–202.

[30] R. Song and Z. Vondracek, Harnack inequality for some discontinuous Markov processes with a diffusion part. Glas. Mat., **40**, (2005), no. 1, 177–187.

[31] K. Yosida, Functional analysis. Springer-Verlag, 6th edition, 1980.